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Analytic Zariski structures and the Hrushovski construction

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Abstract

A set of axioms is presented defining an ‘analytic Zariski structure’, as a generalisation of Hrushovski and Zilber’s Zariski structures. Some consequences of the axioms are explored. A simple example of a structure constructed using Hrushovski’s method of free amalgamation is shown to be a non-trivial example of an analytic Zariski structure. A number of ‘quasi-analytic’ results are derived for this example e.g. analogues of Chow’s theorem and the proper mapping theorem.

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1. Introduction

The aim of the paper is to introduce a formal, model theoretic version of an analytic theory. A similar approach has been used by Hrushovski and Zilber to introduce the notion of a Zariski geometry which thus proved to be a very good and a very useful abstraction of an algebraic–geometric structure or of a compact complex manifold (see [8,14,7] and [11]). A generalisation to a non-compact analytic case, where compactification fails to

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preserve analyticity and brings in essential singularities is a much harder task, yet highly desirable because analytic dimension theory has many parallels with model theoretic stability and a solid link could prove very fruitful. The second author has already made several unsatisfactory attempts to approach the problem (see [15] and [16]) before it was realised that the Hrushovski construction of ‘new strongly minimal structures’ typically produces model examples of what can be seen as formal analytic structures or maybe even genuine analytic structures (see [17]).

The present paper provides a definition of an *analytic Zariski structure* following the pattern of the definition of Zariski structures in [14] and [11]. The leading idea of the present definition is that it attempts to cover a certain type of the new Hrushovski structures. Indeed one of the main results of the paper is the proof that the very first of Hrushovski’s examples in [6] does satisfy the definition. This proves that the definition is indeed a proper extension of the notion of Zariski structure. On the other hand, though we could suggest quite a few very basic genuine analytic structures as candidates for analytic Zariski, e.g. $\mathbb{C}_{\text{exp}} = (\mathbb{C}, +, \times, \text{exp})$, the complex numbers with exponentiation, we do not yet know how to check that they satisfy the definition. Still their *pseudo-analytic* counterparts (see [17]) are much easier to study. So this makes it very intriguing to investigate the relationship between the formal notion of analytic Zariski structure and the genuine analytic theory.

Also, some very important questions remain unanswered, in particular we would like to know under what assumptions the theory of an analytic Zariski structure is stable in the first order or some more general sense.

2. Analytic Zariski structures

The definition of analytic Zariski given here is in no way canonical. It is rather an attempt to bring together some of the analytic-type properties of compact complex manifolds which we believe are characteristic to many of the new structures arising from the Hrushovski construction. There are obviously many variations possible, and these are being explored. Certainly we make no claim that these axioms are independent. Indeed, in the third subsection we show how some fairly natural assumptions on the way that analytic sets are defined render some of them as consequences of the others.

2.1. Definition

A structure \mathbf{M} is said to be a **(compactifiable) analytic-Zariski structure** if there is a structure $\mathbf{P}(\mathbf{M}) = \mathbf{P}$ such that $M \subseteq P$, (where M is the base set of \mathbf{M} and P the base set of \mathbf{P}), \mathbf{P} and \mathbf{M} are inter-definable, and the following conditions A–C hold.

A. [**Language**] There is a collection \mathcal{C} of definable subsets of $\bigcup_{n \in \mathbb{N}} P^n$, called \mathcal{C} -closed sets, such that the following hold:

1. P , any singleton subset of P and the diagonal of $P \times P$ are in \mathcal{C} ;
2. \mathcal{C} is closed under finite unions, finite intersections, Cartesian products and projections;
3. \mathbf{P} is compact with respect to \mathcal{C} : any family of subsets in \mathcal{C} which satisfy the finite intersection property have a non-empty intersection.

A set $S \subseteq P^n$ is called **closed** if $S = \bigcap_{i \in I} C_i$, with $C_i \in \mathcal{C}$, and I any index set. Subsets of P^n of the form $P^n \setminus S$ for a closed S are called **open**. We write $U \subseteq_{op} P^n$ to say that U is open and $S \subseteq_{cl} P^n$ to say that S is closed. If $S \subseteq_{cl} P^n$ then we say that $C = S \cap U$ is closed in U ($C \subseteq_{cl} U$) for any $U \subseteq_{op} P^n$.

4. The subset M is open in P .
5. For $U \subseteq_{op} P^{n+m}$, $C \subseteq_{cl} P^{n+m}$, and $pr : P^{n+m} \rightarrow P^m$ the standard projection map, $pr(U)$ is open in P^m and $pr(C)$ is closed in P^m .

B. [Analytic sets] Given $U \subseteq_{op} P^n$ some subsets of the form $S = C \cap U \subseteq_{cl} U$ are called **analytic in U** . Then we write $S \subseteq_{an} U$ (or abusing notation $C \subseteq_{an} U$) and demand that the following hold:

1. \emptyset , any singleton and U are analytic in U ;
2. If $S_1 \subseteq_{an} U_1$ and $S_2 \subseteq_{an} U_2$, then $S_1 \times S_2$ is analytic in $U_1 \times U_2$;
3. If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cap S_2$ and $S_1 \cup S_2$ are analytic in U ;
4. If $S \subseteq_{an} U$ and $V \subseteq U$ is open then $S \cap V \subseteq_{an} V$;

Let $C \cap U = S \subseteq_{an} U \subseteq_{op} P^{n+m}$ and $pr : P^{n+m} \rightarrow P^n$ be the standard projection map so that $pr(S) \subseteq pr(U) \subseteq_{op} P^n$ (by axiom A.5). We say that the projection is **proper on S** if for any $S' \subseteq S$ such that S' is relatively closed in U we have that $pr(S')$ is closed in $pr(U)$ and for any $a \in pr(S)$ we have that fibre $pr^{-1}(a) \cap S$ is closed in P^{n+m} .

5. If pr is proper on $S \subseteq_{an} U$ then $pr(S)$ is analytic in $pr(U)$;

An analytic set $S \subseteq_{an} U$ is called **(analytic) irreducible** (in U) if there is no $S_1, S_2 \subseteq_{an} U$, $S_i \subsetneq S$, such that $S = S_1 \cup S_2$.

6. If $S \subseteq_{an} U$ and $a \in S$ then there is $S_a \subseteq_{an} U$, a finite union of irreducible analytic subsets of U containing a and some $S'_a \subseteq_{an} U$ such that $a \in S_a \setminus S'_a$ and $S = S_a \cup S'_a$;

Each of the (finite number of) irreducible sets whose union is S_a above is called an **irreducible component of S containing a** .

7. $U \subseteq_{op} P^n$ is irreducible in U .

C. [Dimension] To any non-empty $S \subseteq_{an} U \subseteq_{op} P^n$ a non-negative integer called **the dimension of S** , $\dim S$, is attached.

1. $\dim\{a\} = 0$ for any point $a \in P$, and $\dim(U) = 1$ for any open $U \subseteq P$;
2. If $S_1 \subseteq_{an} U$, $S_2 \subseteq_{an} V$ and $S_1 \subseteq S_2$ then $\dim S_1 \leq \dim S_2$;
3. For an $S \subseteq_{an} U$ $\dim S = \max\{\dim S_a : S_a \text{ irreducible components of } S\}$;
4. If $S \subseteq_{an} U$ is irreducible, V open, then $S \cap V$ is an irreducible analytic subset of V and, if non-empty, $\dim S \cap V = \dim S$;
5. If $S \subseteq_{an} U$ is irreducible, $S_1 \subseteq S$, $S_1 \subseteq_{an} U$, then either $\dim S_1 < \dim S$ or $S = S_1$;
6. Given an irreducible $S \subseteq_{an} U \subseteq_{op} P^{n+m}$ and that the projection $pr : P^{n+m} \rightarrow P^n$ is proper on S we have

$$\dim pr(S) = \dim S - \min\{\dim(pr^{-1}(u) \cap S) : u \in pr(S)\};$$

7. If $S \subseteq_{an} U \subseteq_{op} P^{n+m}$ and $pr : P^{n+m} \rightarrow P^n$ is proper on S then for any k the set

$$\{a \in pr(S) : \dim(pr^{-1}(a) \cap S) \geq k\}$$

is analytic in $pr(U)$.

\mathbf{P} is called the *compactification* of \mathbf{M} , and is called a compact analytic Zariski structure. We also consider the following properties:

[Quantifier Elimination] Any definable subset of P^n is a Boolean combination of subsets of the form $pr(S)$, where $S \subseteq_{an} U \subseteq_{op} P^m$, $m \geq n$, and pr is the projection $P^m \rightarrow P^n$.

[Pre-smoothness] If $S_1, S_2 \subseteq_{an} U$ are both irreducible, then for any irreducible component S_0 of $S_1 \cap S_2$

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim U.$$

[Analytic rank] Let U be an open subset and $S \subseteq U$. The **analytic rank of S in U** $\text{ark}_U(S)$ is a natural number defined by induction as follows:

1. $\text{ark}_U(S) = 0$ iff $S = \emptyset$;
2. $\text{ark}_U(S) \leq k + 1$ iff there is a set $S_1 \subseteq S$, with S_1 closed in U , such that $\text{ark}_U(S_1) \leq k$ and with the set $S_2 = S \setminus S_1$ being analytic in $U \setminus S_1$ (which is open in U and \bar{M}^n);
3. $\text{ark}_U(S) = \min_{n \in \mathbb{N}} \{\text{ark}_U(S) \leq n\}$.

We ask that for any open U and $S \subseteq U$ \mathcal{C} -closed in U there is $n \in \mathbb{N}$ such that $\text{ark}_U(S) = n$.

2.2. General results

We show first that this definition generalises the notion of a Zariski Geometry in [11]. So for this subsection let \mathbf{P} be a compact analytic Zariski structure and \mathcal{C}_0 be the subfamily of \mathcal{C} consisting of subsets analytic in $\bigcup_{n \in \mathbb{N}} P^n$ and \mathbf{P}_0 be the reduction of \mathbf{P} to the language \mathcal{C}_0 .

Theorem 2.2.1. \mathbf{P}_0 is a Zariski structure.

Proof. Comparing our definition with that given in [11], in order to prove the theorem we need only to check the descending chain condition (DCC) for \mathcal{C}_0 and the fact that \mathcal{C}_0 is closed under projections. These are proved in the lemmas below. \square

Proposition 2.2.2. Given $a \in S \subseteq_{an} U \subseteq_{op} P^n$ there is a unique set S_a as described in B6 where the number of irreducible components (of which it is a union) is minimal. Further the (finite number of) irreducible components are also unique.

Proof. Suppose $S_a = S_1 \cup \dots \cup S_k$ and $C_a = C_1 \cup \dots \cup C_k$ are such that S_i and C_i are irreducible for $i = 1, \dots, k$ and there exist S'_a and C'_a as in B6.

First note that $S_i \cap C_a = S_i$ for each i . If not then we have proper inclusion and so get that $S_i \setminus C_a$ is non-empty. Further, as $U \setminus C_a$ is open in U and $S_i \setminus C_a = S_i \cap (U \setminus C_a)$ we get by C4 and the irreducibility of S_i that either $\dim(S_i \setminus C_a) = \dim(S_i)$ or $S_i \setminus C_a = \emptyset$ (but $S_i \setminus C_a$ is non-empty so $\dim(S_i \setminus C_a) = \dim(S_i)$). But since $S_i \subseteq S = C_a \cup C'_a$ we have

$$S_i \setminus C_a = S_i \cap (S \setminus C_a) \subseteq S_i \cap C'_a,$$

so that $\dim(S_i \cap C'_a) \geq \dim(S_i \setminus C_a) = \dim(S_i)$ (by C2). Then by C5 and irreducibility we have $S_i \cap C'_a = S_i$, so that $S_i \subseteq C'_a$. But then $a \notin S_i$ (as $a \in C_a \setminus C'_a$), and this contradicts the minimality of k as we could have omitted S_i from S_a and added it to S'_a .

Thus $S_i \subseteq C_a$ for each i , and this gives that $S_a \subseteq C_a$. Symmetrically we get $C_a \subseteq S_a$, so we have equality.

To show that the components S_i and C_j can be paired off identically we go by induction on k , noting that we have the base case already.

We know that $S_1 \cap (C_1 \cup \dots \cup C_k) = S_1$ and note that we must have $S_1 \subseteq C_j$ for some j by the irreducibility of S_1 , and without loss we assume $j = 1$. Now consider $C_1 \setminus S_1$. Note that since $S_1 \subseteq C_1$ and $S_1 \setminus \bigcup_{i=2}^k S_i \neq \emptyset$ (by minimality of k) we have $C_1 \setminus \bigcup_{i=2}^k S_i \neq \emptyset$ and so

$$C_1 \setminus S_1 = C_1 \cap (U \setminus S_1) \subseteq C_1 \cap (S_2 \cup \dots \cup S_k) \subsetneq C_1.$$

Since $\bigcup_{i=2}^k S_i$ is closed, C5 and the irreducibility of C_1 give $\dim(C_1 \setminus S_1) < \dim(C_1)$. But also $C_1 \setminus S_1 = C_1 \cap (U \setminus C_1)$ and $U \setminus C_1$ is open in U , so by C4 we have either $\dim(C_1 \setminus S_1) = \dim(C_1)$ or $C_1 \setminus S_1 = \emptyset$. Thus $C_1 \setminus S_1 = \emptyset$, so $C_1 \subseteq S_1$ and so we get equality.

By induction on k we are done. \square

Lemma 2.2.3. *Analytic subsets of \mathbf{P}^n have only finitely many irreducible components.*

Proof. Suppose $S \subseteq_{an} \mathbf{P}^n$ has infinitely many components. Then by B6, for any $a \in S$, we have an analytic subset $S'_a \subseteq S$ which does not contain a and contains all but finitely many components of S .

Obviously, the family $\{S'_a : a \in S\}$ has the finite intersection property since for any S'_a, S'_b , $S'_a \cap S'_b$ contains an infinite number of components. Thus by compactness there must be a common point for all members of the family, a contradiction. \square

Lemma 2.2.4. \mathcal{C}_0 satisfies DCC.

Proof. By C2 in any descending \mathcal{C}_0 -chain the dimension will eventually stabilise. By C5 and Lemma 2.2.3 the chain itself eventually stabilises. \square

Lemma 2.2.5. *If $S \in \mathcal{C}_0$ then $pr(S) \in \mathcal{C}_0$.*

Proof. The projection is proper on S , for any $S \in \mathcal{C}$, since for any $a \in pr(S)$ we have that $pr^{-1}(a) \cap S = (\{a\} \times \mathbf{P}^n) \cap S$ is closed in \mathbf{P}^n as an intersection of sets closed there, and A5 gives us that the projections of closed sets are closed. The lemma follows from B5. \square

This finishes the proof of the theorem.

2.3. Natural definitions

Definition 2.3.1. We will say that in an analytic Zariski structure \mathbf{P} the dimension extends to closed subsets if the dimension can be extended to (relatively) closed subsets $S \subseteq_{cl} U \subseteq_{op} \mathbf{P}^n$ in such a way that for any $S' \subseteq_{cl} S$, $\dim S' \leq \dim S$; and for $S_1 \subseteq_{cl} U$, $S_2 \subseteq_{cl} V$ such that $S_1 \cup S_2 \subseteq_{cl} U \cup V$ we have $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$.

Definition 2.3.2. Under the assumption that the dimension extends to closed subsets we say that a closed subset $S \subseteq U$ is **strongly irreducible** if S is relatively definable in U and for any $S' \subseteq_{cl} S$, either $\dim S' < \dim S$ or $S' = S$. We say that a closed (rather than analytic) set S is **(closed) irreducible** if there is no $S_1, S_2 \subseteq_{cl} U$, $S_i \subsetneq S$, such that $S = S_1 \cup S_2$.

Definition 2.3.3. We will say that in analytic Zariski \mathbf{P} (or any structure satisfying the [Language] axioms) **the analytic sets are naturally defined** if

1. the dimension extends to closed subsets in such a way that formula [C6] holds for strongly irreducible sets;
2. strongly irreducible closed sets and irreducible analytic sets coincide; and
3. $S \subseteq_{cl} U \subseteq_{op} \mathbf{P}$ is analytic iff for any $a \in S$ there is an open neighbourhood $a \in V_a \subseteq U$ such that $S \cap V_a$ is the union of finitely many strongly irreducible subsets.

For any $a \in S \subseteq_{cl} U$ such that there is an open neighbourhood $a \in V_a \subseteq U$ such that $S \cap V_a$ is the union of finitely many strongly irreducible subsets, we say that **S is analytic at a** .

- Note 2.3.4.**
1. We note first that though infinite intersections of \mathcal{C} -closed sets may not be \mathcal{C} -closed, our closed sets do form the closed sets of a topology.
 2. If we take our language to be made up of predicates for all the \mathcal{C} -closed sets then some closed sets, and thus analytic sets, may not be definable.
 3. The above, however, means that if the analytic sets are naturally defined and $S \subseteq_{an} U \subseteq_{op} P^n$, then at any point $a \in S$ there is an open $V_a \ni a$ such that S is a finite union of sets which are strongly irreducible, and so definable, inside V_a . So analytic sets are locally definable.
 4. A closed $S \subseteq U$ which is strongly irreducible is a union of a single strongly irreducible set at each of its points, and thus is analytic under a natural definition. We also notice here that strong irreducibility implies irreducibility as a set cannot be the union of two sets of strictly lower dimension (see Proposition 2.3.7). So, under natural definitions, strong irreducibility implies analytic and closed irreducibility, and analytic irreducibility implies strong irreducibility.
 5. By C5 we get the main feature of strong irreducibility for analytic irreducible sets, but, without the assumption of natural definitions, we cannot guarantee global definability of irreducible sets as is required for strong irreducibility.
 6. With the assumption of natural definitions we get that [C6] holds for all irreducible sets automatically, and thus all analytic irreducible sets also.

We now show, under the assumption that the analytic sets are naturally defined (or even weaker assumptions), that some of the axioms are redundant. Thus, unless stated otherwise, we assume until the end of the section we have the [Language] axioms in \mathbf{P} and also that the analytic sets are naturally defined. We state exactly which of the other axioms we assume in the proof of each result.

Lemma 2.3.5. *Assuming only that dimension extends to closed subsets we have that if $S \subseteq_{cl} U \subseteq_{op} P^n$, S is strongly irreducible in U and $V \subseteq_{op} U$ then $S \cap V \subseteq_{cl} V$ is strongly irreducible in V .*

Proof. Suppose that S and V are as described, and note that $U \setminus V$ is closed in U . Now suppose that $S \cap V \neq \emptyset$ (\emptyset is trivially strongly irreducible) so that $S \setminus V \subsetneq S$. This gives us that $S \setminus V = S \cap (U \setminus V)$ is a proper closed subset of S and so $\dim(S \setminus V) < \dim(S)$. But then $\dim(S) = \max\{\dim(S \setminus V), \dim(S \cap V)\}$ give us that $\dim(S) = \dim(S \cap V)$. Also note that for any $S_1 \subsetneq S$ closed in U we have $\dim(S_1) < \dim(S)$ and so for any $S_1 \cap V \subsetneq S \cap V$ closed in V we have $\dim(S_1 \cap V) \leq \dim(S_1) < \dim(S) = \dim(S \cap V)$. So $S \cap V$ is strongly irreducible. \square

Lemma 2.3.6 (B4). *Assuming only that dimension extends to closed subsets and the third part of the statement of natural definitions we have that if $S \subseteq_{an} U$ and $V \subseteq U$ is open then $S \cap V \subseteq_{an} V$.*

Proof. Let $a \in S \cap V$. Then $a \in S$ and so by analyticity there is $V_a \ni a$ open in U such that $S \cap V_a$ is a finite union of sets strongly irreducible in V_a . But then, by Lemma 2.3.5, $(S \cap V) \cap V_a$ is a finite union of sets strongly irreducible in $V \cap V_a$. So $S \cap V$ is analytic at a , and so analytic in V as a was arbitrary. \square

Proposition 2.3.7 (C5). *If $S \subseteq_{an} U \subseteq_{op} P^n$ then S is irreducible in U iff for any $S_1 \subsetneq S$, $S_1 \subseteq_{cl} U$, we have $\dim S_1 < \dim S$.*

Proof. The right to left direction is obvious since if all proper closed subsets of S are of lower dimension, then any union of 2 proper closed subsets is also of lower dimension, and similarly for any proper analytic subsets. The left to right direction follows from natural definitions since they give that S is strongly irreducible. \square

Lemma 2.3.8. *If $S \subseteq_{an} U \subseteq_{op} P^n$ and $S = C_1 \cup C_2$ is such that $C_i \subseteq_{cl} U$ and $C_i \subsetneq S$ then there are $S_1, S_2 \subseteq_{an} U$ such that $S = S_1 \cup S_2$ and $S_i \subsetneq S$.*

Proof. By natural definitions dimension extends to closed subsets and so $\dim(S) = \max\{\dim(C_1), \dim(C_2)\}$. But as these are proper subsets this means that S is not strongly irreducible, and thus by natural definitions not analytic irreducible. \square

Proposition 2.3.9 (C4). *If $S \subseteq_{an} U \subseteq_{op} \bar{M}^n$ is such that S is irreducible, and $V \subseteq_{op} U$ then $S \cap V$ is irreducible in V , and if it is non-empty then $\dim(S \cap V) = \dim(S)$.*

Proof. By natural definitions we then get that S is strongly irreducible and so by 2.3.5 $S \cap V$ is strongly irreducible in V and so analytic irreducible in V .

Since S is strongly irreducible and noting that $C = U \setminus V$ is closed in U we get that, if $S \cap V \neq \emptyset$, so that $S \cap C \subsetneq S$ then $\dim(S \setminus V) = \dim(S \cap C) \leq \dim(S)$. But this means that $\dim(S) = \dim((S \cap V) \cup \dim(S \setminus V)) = \dim(S \cap V)$, as required. \square

Note 2.3.10. It turns out that we will need these last three results in order to get that the analytic sets are naturally defined in the Hrushovski structure, so we will have to reprove them without the assumption that analytic sets are naturally defined, but only in the specific case of the Hrushovski structure (see 5.2.8).

Remark 2.3.11. If $S \subseteq_{an} U \subseteq_{op} P^n$ is irreducible but not definable then we can conclude that S is locally strongly irreducible at every point.

Proof. Take any $a \in S$, and note that by analyticity there is in open $V_a \ni a$ such that $S \cap V_a = S_1 \cup \dots \cup S_k$, with S_i strongly irreducible in V_a . We want to show that $k = 1$. If $k > 1$ non-trivially then say that $S_i = C_i \cap V_a$ for each i with $C_i \subseteq_{cl} P^n$. Then, letting $S = C \cap U$ for $C \subseteq_{cl} P^n$ and noting that $C' = U \setminus C$ is closed in U we have that $S = C' \cup \bigcup_{i=1}^k (C \cap C_i)$ and this union is non-trivial since it is when restricted to V_a . But this contradicts S being irreducible by Lemma 2.3.8. \square

Lemma 2.3.12. *If $S \subseteq_{cl} W \subsetneq_{op} P^n$ and $C \subseteq W$ is such that C is closed in P^n , then $C \cap S$ is closed in P^n .*

Proof. Say $S = S_c \cap W$ where $S_c \subseteq_{cl} P^n$. Then $C \cap S = C \cap S_c \cap W = C \cap S_c$ is closed in P^n . \square

Lemma 2.3.13. *Assume that B6 holds. Let $S \subseteq_{an} W \subsetneq_{op} P^n$ and $C \subseteq S$ be such that $C \subseteq_{cl} P^n$. Then there are S_1, \dots, S_k such that each S_i is analytic irreducible in W and $S' \subseteq_{cl} W$ such that $S = \bigcup_{i=1}^k S_i \cup S'$ and $C \cap S' = \emptyset$.*

Proof. First note that for any $a \in C$ we have $a \in S$, so by the analyticity of S and B6 there is S_a , a finite union of sets analytic irreducible in W , and $S'_a \subseteq_{an} W$ such that $S = S_a \cup S'_a$ and $a \notin S'_a$. Consider $\bigcap \{S'_a \mid a \in C\}$. For any $a' \in C$, $a' \notin S'_a$ and so $a' \notin \bigcap \{S'_a \mid a \in C\}$. Thus $C \cap \bigcap \{S'_a \mid a \in C\} = \emptyset$ i.e. $\bigcap \{(C \cap S'_a) \mid a \in C\} = \emptyset$. Now since $C \subseteq S$ and $C \subseteq_{cl} P^n$ we have that $C \cap S'_a \subseteq_{cl} P^n$ by 2.3.12, and then by compactness (A3) we have that there must be an empty finite sub-intersection. Say $a_1, \dots, a_k \in S$ are such that $\bigcap_{i=1}^k (C \cap S'_{a_i}) = \emptyset$ so that, writing $S' = \bigcap_{i=1}^k S'_{a_i}$, we get $C \cap S' = \emptyset$. Also, writing S_i and S'_i for S_{a_i} and S'_{a_i} respectively we note $S \setminus S_i \subseteq S'_i$ and so

$$\begin{aligned} S &= \bigcup_{i=1}^k S_i \cup \left(S \setminus \bigcup_{i=1}^k S_i \right) \\ &= \bigcup_{i=1}^k S_i \cup \bigcap_{i=1}^k (S \setminus S_i) \\ &\subseteq \bigcup_{i=1}^k S_i \cup \bigcap_{i=1}^k S'_i \\ &= \bigcup_{i=1}^k S_i \cup S' \subseteq S. \end{aligned}$$

And so we get equality throughout. Since each $S_i = S_{a_i}$ is a finite union of sets irreducible in W this gives the result. \square

We are now ready to show that under the assumption that analyticity is defined naturally the axiom B5 is redundant.

Theorem 2.3.14 (B5). *Assume that B6 holds. Given $S \subseteq_{an} W \subsetneq_{op} P^n$ and $pr : P^n \rightarrow P^m$ a standard projection such that $pr(S) \subseteq U \subseteq_{op} P^m$, suppose pr is proper on S . Then $pr(S)$ is analytic in U .*

Proof. Say $a \in pr(S)$ and note that by properness $S_a := pr^{-1}(a) \cap S \subseteq_{cl} P^n$. Also $S_a \subseteq S$ and so by Lemma 2.3.13 there are sets, S_1, \dots, S_k , analytic irreducible in W and $S' \subseteq_{an} W$ such that $S_a \cap S' = \emptyset$ and $S = \bigcup_{i=1}^k S_i \cup S'$. Now $S_a \cap S' = pr^{-1}(a) \cap S' = \emptyset$ and so $a \notin pr(S')$. So putting $U_a = U \setminus pr(S') \subseteq_{op} P^m$ we get $a \in U_a$. By properness each $pr(S_i)$ is closed in U . Since each S_i is analytic irreducible, by natural definitions we have that they are all strongly irreducible. We get from this that each $pr(S_i)$ is also strongly irreducible. For if it weren't then there would be some closed $C \subsetneq pr(S_i)$ with $\dim(C) = \dim(pr(S_i))$. Then by C6, which we have for strongly irreducible sets since we are assuming that the analytic sets are naturally defined, we would have:

$$\begin{aligned} \dim(S_i) &= \dim(pr(S_i)) + \min_{a \in pr(S_i)} (\dim(pr^{-1}(a) \cap S_i)) \\ &\leq \dim(C) + \min_{a \in C} (\dim(pr^{-1}(a) \cap S_i)) \\ &\leq \dim((C \times P^{n-m}) \cap S_i), \end{aligned}$$

and since $(C \times P^{n-m}) \cap S_i \subsetneq S_i$, this contradicts the strong irreducibility of S_i . Thus

$$\begin{aligned} pr(S) \cap U_a &= pr \left(\bigcup_{i=1}^k S_i \cup S' \right) \cap (U \setminus pr(S')) \\ &= \left(\bigcup_{i=1}^k pr(S_i) \cup pr(S') \right) \cap (U \setminus pr(S')) \\ &= \bigcup_{i=1}^k pr(S_i) \cap (U \setminus pr(S')) \\ &= \bigcup_{i=1}^k (pr(S_i) \cap U_a), \end{aligned}$$

which is a finite union of strong irreducibles. Thus, by natural definitions, $pr(S)$ is analytic at a . \square

Lemma 2.3.15. Assume that B3, B6, C3, C5 and C6 hold. Let $V \subseteq P^n$ be an open subset and

$$\{T^b : b \in B\}$$

any family of analytic subsets $T^b \subseteq_{an} V$. Then

$$T^* = \bigcap_{b \in B} T^b \subseteq_{an} V$$

(i.e. an infinite intersection of analytic sets is analytic).

Proof. Suppose $a \in T^*$. Then, for any $b_1, \dots, b_k \in B$ we have by B3 that $T^{b_1} \cap \dots \cap T^{b_k}$ is analytic and so there are finitely many irreducible components of $T^{b_1} \cap \dots \cap T^{b_k}$ which contain a . Let $(T^{b_1} \cap \dots \cap T^{b_k})_a$ be the union of these components and choose b_1, \dots, b_k , depending on a , first so that the number of the components is minimal and then so that the

dimension of each of them are the minimal possible, and note that this is not necessarily when $k = 1$. Then by C3

$$(T^{b_1} \cap \dots \cap T^{b_k})_a = (T^{b_1} \cap \dots \cap T^{b_k})_a \cap T^*,$$

since if not we could find $b_{k+1} \in B$ such that $(T^{b_1} \cap \dots \cap T^{b_k} \cap T^{b_{k+1}})_a \subsetneq (T^{b_1} \cap \dots \cap T^{b_k})_a$, and this means by C3 that either $(T^{b_1} \cap \dots \cap T^{b_k} \cap T^{b_{k+1}})_a$ has fewer irreducible components, or components of lower dimension, either of which contradicts the minimality of our choice of b_1, \dots, b_k .

We can now find, by B6, a subset $(T^{b_1} \cap \dots \cap T^{b_k})'_a$, analytic in V , which does not contain a and such that

$$(T^{b_1} \cap \dots \cap T^{b_k})_a \cup (T^{b_1} \cap \dots \cap T^{b_k})'_a = (T^{b_1} \cap \dots \cap T^{b_k}).$$

Let

$$V_a = V \setminus (T^{b_1} \cap \dots \cap T^{b_k})'_a.$$

Then

$$T^* \cap V_a = (T^{b_1} \cap \dots \cap T^{b_k})_a \cap V_a,$$

that is T^* in the neighbourhood is equal to a finite union of irreducible sets.

If $a \notin T^*$ then there is $b \in B$ such that $a \notin T^b$. Putting $V_a = V \setminus T^b$ we have $a \in V_a$ and clearly $T^* \subseteq T^b$ so that $T^* \cap V_a = \emptyset$, the empty union of sets irreducible in V_a . This gives us that T^* is analytic at a . \square

Notation 2.3.16. Let $pr_{n,m}$ denote the standard projection map from $n + m$ space (P^{n+m}) to n space (P^n). We note that then $pr_{n+1,m-1}$ maps $n + m$ space to $n + 1$ space.

Lemma 2.3.17. *If $pr : P^{m+n} \rightarrow P^n$ is proper on $S \subseteq_{cl} U$ then it is also proper on any $S' \subseteq S$ with $S' \subseteq_{cl} U$.*

Proof. Firstly, for any $C \subseteq S'$ such that C is closed in U we have that $C \subseteq S$ and is closed in U , so by properness on S , $pr(C)$ is closed in $pr(U)$. Also, for any such $S' \subseteq S$ we have $S' = S \cap T$ for some $T \subseteq_{cl} P^{m+n}$ and for any $a \in pr(S')$ a is in $pr(S)$ and so $pr^{-1}(a) \cap S \subseteq_{cl} P^{m+n}$ by properness on S . But then $pr^{-1}(a) \cap S' = pr^{-1}(a) \cap S \cap T$ is closed in P^{n+m} as an intersection of closed sets. \square

Lemma 2.3.18. *Assume that B6, B7, C1, C5 and C6 hold. For any $S \subseteq_{cl} P^m$ we have that $\dim(S) \geq k$ iff there is a projection $pr_{i_1 \dots i_k} : P^m \rightarrow P^k$ along some k of the m coordinates such that $pr_{i_1 \dots i_k}(S) = P^k$.*

Proof. The right to left direction is clear, so let us assume that $\dim(S) \geq k$ and show the existence of $pr_{i_1 \dots i_k} : P^m \rightarrow P^k$ such that $pr_{i_1 \dots i_k}(S) = P^k$. We note that we can assume that S is strongly irreducible (and so irreducible) by taking a minimal subset of equal dimension.

Let $pr_1 : P^m \rightarrow P$ be the projection onto the first co-ordinate. Note that P is analytic irreducible of dimension 1 by B1, B7 and C1, and that, when working with natural definitions, strong irreducibility and irreducibility coincide. Then $pr_1(S)$ is closed (by A5) and is of dimension 0 or equal to P by strong irreducibility.

In the first case by the addition formula (C6, which we have by Note 2.3.4 part 6) $\dim(pr_1^{-1}(b) \cap S) \geq k$ for any $b \in pr_1(S)$, and since $pr_1^{-1}(b) \cap S \subseteq \{b\} \times P^{m-1}$ we can use induction on m to get $pr_{i_1 \dots i_k} : P^{m-1} \rightarrow P^k$ with full image. Note that the base for induction goes through since if $m = 1$ then $k = 1$ or $k = 0$ and the claim clearly holds.

So we assume that $pr_1(S) = P$. Then by the addition formula (C6) again we get that $\dim(pr_1^{-1}(b) \cap S) \geq k - 1$ for all $b \in P$. Since S and $\{b\} \times P^{m-1}$ are both closed in P^m so is their intersection $(pr_1^{-1}(b) \cap S)$ and its projection so that $pr_1^{-1}(b) \cap S = \{b\} \times S^b$ for some closed $S^b \subseteq_{cl} P^{m-1}$. Again by induction on m we have for any such b the existence of $pr_{i_1 \dots i_{k-1}} : P^{m-1} \rightarrow P^{k-1}$ with $pr_{i_1 \dots i_k}(S^b) = P^{k-1}$ (note that $i_1 \dots i_k$ depend on b). Thus

$$pr_{1, i_1 \dots i_{k-1}}(\{b\} \times S^b) = \{b\} \times P^{k-1}.$$

This implies $\{b\} \times P^{k-1} \subseteq pr_{1, i_1 \dots i_{k-1}}(S)$ and so for any $\bar{c} \in P^{k-1}$ we have $\langle b, \bar{c} \rangle \in pr_{1, i_1 \dots i_{k-1}}(S)$. Hence

$$b \in \bigcap_{\bar{c} \in P^{k-1}} pr_1(\{\bar{x} \in P^m : x_{i_1} = c_{i_1}, \dots, x_{i_{k-1}} = c_{i_{k-1}}\} \cap S) = L_{i_1 \dots i_{k-1}}.$$

This $L_{i_1 \dots i_{k-1}}$ is a closed subset of P , as it is an intersection of closed sets. There are only finitely many choices for $i_1 \dots i_k$ (out of $\{2, \dots, n\}$) and thus the $L_{i_1 \dots i_{k-1}}$ form a finite cover of the whole of P . By its irreducibility P must be equal to one of the $L_{i_1 \dots i_{k-1}}$. This implies that $pr_{1, i_1 \dots i_{k-1}}(S) = P^k$. \square

Proposition 2.3.19 (C7). *Assuming that all the B axioms, and C1, C3, C5 and C6 hold, given a projection $pr : P^{n+m} \rightarrow P^n$ which is proper on $S \subseteq_{an} U \subseteq_{op} P^{n+m}$, we have that for any k the set*

$$\{a \in pr(S) : \dim(pr^{-1}(a) \cap S) \geq k\}$$

is analytic in $pr(U) \subseteq_{op} P^n$.

Proof. By properness of the projection the fibre $pr^{-1}(a) \cap S$ over a is a closed subset of $\{a\} \times P^m$, and so it is a subset of the form $pr^{-1}(a) \cap S = \{a\} \times S_a$ for $S_a \subseteq P^m$ closed.

Assuming that $\dim(pr^{-1}(a) \cap S) \geq k$, by Lemma 2.3.18 above we have a projection $pr_{i_1 \dots i_k} : P^m \rightarrow P^k$ such that $pr_{i_1 \dots i_k}(S_a) = P^k$ (with i_1, \dots, i_k dependent on a).

Then, for any $b = \langle b_1, \dots, b_k \rangle \in P^k$

$$\{b_1 = x_{i_1} \wedge \dots \wedge b_k = x_{i_k}\} \cap S$$

is analytic in U as it is an intersection of sets analytic in U . Denote

$$T_b = pr(\{b_1 = x_{i_1} \wedge \dots \wedge b_k = x_{i_k}\} \cap S),$$

which is an analytic subset of $pr(U)$, as the proper projection of an analytic set by Lemma 2.3.14.

Hence

$$R_{i_1 \dots i_k} = \bigcap_{b \in P^k} T_b$$

is analytic in U , by Lemma 2.3.15, as is

$$R = \bigcup \{R_{i_1 \dots i_k} \mid \langle i_1, \dots, i_k \rangle \text{ is a } k\text{-tuple of distinct elements of } \{1, \dots, n\}\},$$

by B3.

But $a' \in R$ if and only if for all $b \in P^k$ we have that $b \in pr_{i_1 \dots i_k}(pr^{-1}(a') \cap S)$ for some choice of i_1, \dots, i_k . i.e. $a' \in R$ if and only if $pr_{i_1 \dots i_k}(pr^{-1}(a') \cap S) = P^k$. So by Lemma 2.3.18 we have that $a' \in R$ if and only if $\dim\{pr^{-1}(a') \cap S\} \geq k$, and thus we are done. \square

Note 2.3.20. Under the assumption of the natural definition of analytic sets, axioms [B4], [B5], [C4], [C5], [C6] and [C7] follow immediately from the others.

Proof. The definition of natural definitions gives [C6] immediately, and then the other axioms come from 2.3.6, 2.3.14, 2.3.9, 2.3.7 and 2.3.19. \square

3. The basic Hrushovski structure

We consider here a simplified version of the structure introduced in 1993 by Hrushovski in his paper [6] which provided a counter-example to Zilber's conjecture on the geometry of infinite-dimensional minimal sets. The respect in which it is simpler is that we do not restrict the number of elements realising each type, as Hrushovski does with his μ -function. This alteration means that our structure is not strongly minimal, but only ω -stable, and of rank ω .

The structure is constructed as a Fraissé amalgam of finite structures described in a language with one ternary predicate, R , obeying one simple axiom which asserts the non-negativity of a pre-dimension function. Pre-dimensions and their non-negativity are the pivotal ideas in the construction of all the counter-examples to Zilber's conjecture, which we shall refer to here as Hrushovski-type structures. Zilber has observed that these Hrushovski-type structures are, viewed in a certain way, quite similar to classical analytic structures. His observation came initially from the fact that, as can be seen in [17], the non-negativity of a pre-dimension function can be seen to be interpretable as a generalised form of the Schanuel Conjecture. We show later that the simplified Hrushovski structure that we study is an analytic Zariski structure.

Normally M, M', \bar{M} , etc. are models, A, B etc. are sets, u, v, w, x, y, z are variables, or finite tuples of variables if barred, and a, b, c are parameters, finite tuples of parameters if barred. We use $|\bar{x}|$ to mean the number of variables in the tuple of variables \bar{x} . $|\bar{c}|$ means the number of distinct elements in the tuple of parameters \bar{c} . $c \in \bar{c}$ means the element c is in the tuple \bar{c} . $\bar{c}' \subseteq \bar{c}$, means that \bar{c}' is a sub-tuple.

In this section we freely refer to results from [18,5], and [3], but all the results are either implicit or explicit in [6].

3.1. Definitions

We work in the language $\mathcal{L} = \{R\}$, where R is interpreted as a ternary relation, which we call the Hrushovski relation or H-relation. Hrushovski's structure is constructed using a pre-dimension function, δ , on finite subsets, where:

$$\delta(X) = |X| - r(X)$$

where $|X|$ is the (finite) size of X and $r(X)$ is the number of distinct triples $\langle a_1, a_2, a_3 \rangle$ from X such that $R(\langle a_1, a_2, a_3 \rangle)$ holds. We then consider the class of structures: $\mathcal{K} = \{A \mid A \text{ is an } \mathcal{L}\text{-structure and } \delta(A') \geq 0 \text{ for any finite } A' \subseteq A\}$.

We define a relative dimension function $d_B(A)$, the dimension of A inside B , when A is finite, as $d_B(A) = \min\{\delta(A \cup X) \mid X \subseteq_{fin} B\}$. We then call A **strong** in B if $\delta(A) = d_B(A)$, and write $A \leq B$. Note that $M \in \mathcal{K}$ iff $\emptyset \leq M$.

Definition 3.1.1. • If A, B are finite let $\delta(A/B) = \delta(A \cup B) - \delta(B)$;

• If $B \subseteq M$ is infinite we let $\delta(A/B) = \min\{\delta(A/B') : B' \subseteq_{fin} B\}$;

• Then for finite A and any B , both in some model M ,

$$d_M(A/B) = \min\{d_M(AB') - d_M(B') : B' \subseteq_{fin} M\}.$$

Note 3.1.2. With these definitions, for infinite $A \subseteq M$, we have $A \leq B$ iff for every $X \subseteq_{fin} B$ we have $\delta(X/A) \geq 0$.

We also note here that for any elementary extension $M' \succeq M$ we have that $M \leq M'$. For if not then there is a finite $X \subseteq M'$ such that $\delta(X/M) < 0$, but then, since M' is an elementary extension, there is a finite $X \subseteq M$ such that $\delta(X/M) < 0$. This is a contradiction since for any $X \subseteq_{fin} M$ we have $\delta(X/M) = 0$. Hence, for any finite $A \subseteq M$ we have $d_M(A) = d_{M'}(A)$.

Due to completeness, given $A \subseteq_{fin} M$, for any model M' of $T \cup \text{edig}(A)$ (where $\text{edig}(A)$ is the elementary diagram of A) we get that $d_{M'}(A) = d_M(A)$, and so we can drop the subscript M in d_M .

Proposition 3.1.3. For finite X and W , and any A we have

$$\delta((W \cup X)/A) = \delta(W/(A \cup X)) + \delta(X/A).$$

Lemma 3.1.4. For finite a and b , and any A , all subsets of a model M

$$d_M(ab/A) = d_M(b/(Aa)) + d_M(a/A).$$

Proof. See [5]. \square

3.2. Construction and axiomatisation

We consider a Fraissé amalgam of all structures which are in the class:

$$\mathcal{K}_0 = \{A \mid A \text{ is a finite } \mathcal{L}\text{-structure and } \phi \leq A\} / \sim$$

where ϕ is the empty set, and the equivalence relation \sim is isomorphism.

We only get the amalgamation condition from Fraissé's Theorem if we restrict our attention to those embeddings $f : A \rightarrow B$ where $f(A) \leq B$. We thus use Theorem 2.10 of [3] to get our amalgamation.

With this restriction we have the Amalgamation Property, the Hereditary Property and the Joint Embedding Property in \mathcal{K}_0 for strong embeddings, and so, by Theorem 2.10 of [3] there is an \mathcal{L} -structure, M , such that

1. M is countable;
2. The class of strongly embedded structures of M is equal to \mathcal{K}_0 ;
3. If $A \leq M$ and $f : A \rightarrow B$ is a strong embedding then there is $B' \leq M$ containing A and an isomorphism $g : B \rightarrow B'$ such that $gf = id_A$;
4. Any isomorphism between strongly embedded finite substructures of M extends to an automorphism of M ;
5. M is unique up to isomorphism.

We do not, however, get \aleph_0 -categoricity, as the uniform boundedness condition of the theorem is not satisfied. To see this we note that the only way to guarantee that every embedding of some $A \in \mathcal{K}_0$ into another element of \mathcal{K}_0 is strong is to have $\delta(A) = 0$. But given $C \subseteq D \in \mathcal{K}_0$ with $\delta(C) = d(C : D) = k > 0$, there may not even be any $A \in \mathcal{K}_0$ containing C with $\delta(A) = 0$. So the size of such an A is certainly not given as a function of the size of C .

We now wish to find some axioms characterising this type of structure. In our constructed structure we clearly have:

Axiom 1. Every finite substructure of M is in \mathcal{K}_0 .

This can be expressed as an infinite number of first-order sentences—for each $n \in \mathbb{N}$ write $\forall x_1, \dots, x_n \, r(x_1, \dots, x_n) \leq n$.

We also have what could be called existential closedness under strong extensions of strong subsets, i.e.:

Fact 3.2.1. For any finite $A \subseteq M$ such that $A \leq M$, and any finite $B \geq A$ there is an isomorphism $\pi : B \rightarrow \hat{B} \subseteq M$ such that $\pi|_A = id_A$ and $\hat{B} \leq M$.

But this is not expressible as a set of first order axioms since to express the relation ' $X \leq M$ ' we need to negate an infinite number of possible H-relations with members of M (if $\delta(X) > 0$).

We state an axiom scheme which is equivalent to the above fact in a saturated model:

Axiom 2. For any finite $A \subseteq M$ and finite $B \geq A$ there is an isomorphism $\pi : B \rightarrow \hat{B} \subseteq M$ such that $\pi|_A = id_A$.

To write this as a set of first order sentences we need to work out every finite configuration of elements B which make $B \geq A$ (where, by configuration we mean specifically what H-relations they could satisfy) and then form an existential sentence asserting the existence of such a B .

It is equivalent to the fact since in a saturated model of **Axioms 1** and **2** we get that there is an infinite d_M -independent set (see **Definition 3.4.3** which then gives us the fact).

Notation 3.2.2. • We refer to the triple relation R as the H-relation;

- We let $r(X \leftrightarrow Y)$ denote the number of distinct H-relations involving elements of both $X \setminus Y$ and $Y \setminus X$;
- We write AX for $A \cup X$ and Aa for $A \cup \{a\}$.

Definition 3.2.3. The theory T consists of all instances of the axiom schemes [Axioms 1](#) and [2](#).

3.3. Basic results

We present some results which are known for structures constructed by Hrushovski's method.

Theorem 3.3.1. T is complete.

Proof. By reference to other similar theories, including those developed in [9] and [18], we see that we can play an Ehrenfeucht–Fraïssé game between any two sufficiently saturated models of T , proving elementary equivalence. In fact, since we only need consider types over finite sets of parameters, ω -saturated models are sufficiently saturated, and we can always extend our models to ω -saturated elementary extensions. Thus any two models of T are elementary equivalent. \square

Theorem 3.3.2. The theory T has quantifier elimination up to Boolean combinations of existential formula.

Proof. This is also standard in this type of theory. See, for example, [5] or [18]. \square

Corollary 3.3.3. If $M \models T$, $A \leq M$, and $\bar{a}, \bar{b} \in M^n$ are such that $A\bar{a} \leq M$ and $A\bar{b} \leq M$ then the following are equivalent

1. $\text{qftp}_{\mathcal{L}}(\bar{a}/A) = \text{qftp}_{\mathcal{L}}(\bar{b}/A)$;
2. \bar{a} and \bar{b} are conjugated by some A -automorphism of M ;
3. $\text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A)$.

Proof. Clearly $2 \Leftrightarrow 3$, and $3 \Rightarrow 1$. We also know, using the arguments from the proofs of completeness, that 1 implies that the partial A -automorphism from $A\bar{a}$ to $A\bar{b}$ can be extended in this case to a full automorphism, which gives us 2 and so we are done. \square

Theorem 3.3.4. T is ω -stable.

Proof. Follows from 3.3.3 by directly counting, for each n , the quantifier free n -types over $A \leq M$, in the same way as in [18]. \square

Remark 3.3.5. In fact the Morley Rank of $x = x$ in this structure is ω .

Definition 3.3.6. A structure M is **atomic compact** iff every set, Φ , of positive primitive formulas with parameters which is finitely realised in M is realised in M . A theory is atomic compact iff all of its models are.

Lemma 3.3.7. M is atomic compact iff for every elementary extension $M' \succeq M$ there is a homomorphism $f: M' \rightarrow M$ such that $f|_M = \text{id}_M$

Proof. See [4] or [13]. \square

We are interested in this property because it was the first hint that the Hrushovski structure might have a topological meaning. Although we will not eventually use atomic compactness in this simple language we include the proof here for completeness.

Theorem 3.3.8. *T is atomic compact.*

Proof. Take an arbitrary model of T , M , and an arbitrary elementary extension $M' \succeq M$. We show the existence of a homomorphism $f: M' \rightarrow M$ such that $f|_M = id_M$ by induction on the size of its domain.

Let $f(m) = m$, $\forall m \in M$ and assume we've already extended f to $f: D \rightarrow M$ a homomorphism where $M \subseteq D \subseteq M'$ and $D \leq M'$.

Choose $m \in M' \setminus D$. We extend m to a tuple \bar{m} with elements from $M' \setminus D$ such that $D\bar{m} \leq M'$.

If $d_{M'}(m/D) = 0$ take a minimal extension \bar{m} such that $\delta(\bar{m}/D) = 0$. For any $X \subseteq_{fin} M'$ we get that $\delta(X\bar{m}/D) \geq 0$ since $X\bar{m} \subseteq_{fin} M'$, and $D \leq M'$. From this we also get that $\delta(X/D\bar{m}) = \delta(X\bar{m}/D) - \delta(\bar{m}/D) = \delta(X\bar{m}/D) \geq 0$ and so $\delta(X/D\bar{m}) \geq 0$ for all $X \subseteq_{fin} M'$, i.e. $D\bar{m} \leq M'$.

If $d_{M'}(m/D) = 1$ then let $\bar{m} = m$. We have that $\delta(Xm/D) \geq 1$ for any $X \subseteq_{fin} M'$, and clearly $\delta(m/D) = 1$. Thus for any $X \subseteq_{fin} M'$ we have:

$$\delta(X/Dm) = \delta(Xm/D) - \delta(m/D) \geq 1 - 1 = 0.$$

Thus $D\bar{m} \leq M'$.

By definition of d_D , for any finite $A_0 \subseteq D$ we can find a finite $A \subseteq D$, $A_0 \subseteq A$, such that $\delta(A) = d_D(A_0) = d_D(A) = d_{M'}(A)$, the final equality holding since we have $M \leq M'$ from $M \preceq M'$. For any finite $X \subseteq M'$, we have $\delta(A) \leq \delta(AX)$ (since $D \leq M'$) and so for any sub-tuple $\bar{m}' \subseteq \bar{m}$ we have

$$\delta(A) \leq \delta(A\bar{m}') = |A| + |\bar{m}'| - r(A) - r(\bar{m}') - r(\bar{m}' \leftrightarrow A)$$

which gives $|\bar{m}'| - r(\bar{m}') \geq r(\bar{m}' \leftrightarrow A)$ and hence $|\bar{m}'| \geq r(\bar{m}' \leftrightarrow D)$ (since if \bar{m}' was involved in more positive relations with elements of D then we could choose a finite A_0 such that $r(\bar{m}' \leftrightarrow A_0) > |\bar{m}'|$) a contradicting $|\bar{m}'| - r(\bar{m}') \geq r(\bar{m}' \leftrightarrow A)$.

Say \bar{m} is in H-relations with $A \subseteq_{fin} D$, and no other elements of D . By the argument above we can assume that $A \leq M'$. Then $A \leq A\bar{m}$, since $\bar{m} \subseteq_{fin} M'$. We wish to find an homomorphic image of $A\bar{m}$ inside M .

Consider $\text{qftp}(\bar{m}/D)$. This only contains $\leq |\bar{m}|$ H-relations. Replacing occurrences of any $a \in A$ in $\text{qftp}(\bar{m}/D)$ by its image under f ($f(a) \in M$) we get $f(\text{qftp}(\bar{m}/D))$ (note that this is $\neq \text{qftp}(\bar{m}/f(D)) = \text{qftp}(\bar{m}/M)$). Now, given any \bar{m}_0 (not necessarily in a model of T) which satisfies $f(\text{qftp}(\bar{m}/D))$ and no extra H-relations, and noting that $\{x_i \neq x_j\}_{i \neq j} \subseteq \text{qftp}(\bar{m}/D)$ and letting \bar{m}'_0 be any sub-tuple of \bar{m}_0 we get:

$$\begin{aligned} |\bar{m}'_0| - r(\bar{m}'_0) &\geq r(\bar{m}'_0 \leftrightarrow f(A)) \\ \text{i.e.} \quad |\bar{m}'_0| &\geq r(f(A)\bar{m}'_0) - r(f(A)) \\ \text{i.e. } |f(A)| + |\bar{m}'_0| - r(f(A)\bar{m}'_0) &\geq |f(A)| - r(f(A)) \\ \text{i.e. } \delta(f(A)) &\leq \delta(f(A)\bar{m}'_0). \end{aligned}$$

And thus we have that $f(A)$ is strong in $f(A)\bar{m}_0$ (i.e. $f(A) \leq f(A)\bar{m}_0$) for any such \bar{m}_0 .

Also $f(A) \subseteq M$ and so by Axiom 2 there is $\hat{m} \in M$ and an isomorphism $\pi: f(A)\bar{m}_0 \rightarrow f(A)\hat{m}$ such that $\pi|_{f(A)} = id_{f(A)}$. Extend f by making $f(\bar{m}) = \pi(\bar{m}_0) = \hat{m}$. Note that all positive relations are preserved since any positive formula with parameters \bar{a} from D ,

$\psi(\bar{x}, \bar{a})$, such that $M' \models \psi(\bar{m}, \bar{a})$ will only involve $\leq |\bar{m}|$ H-relations, and the parameters \bar{a} are from A . We get that \bar{m}_0 satisfies $\psi(\bar{x}, f(\bar{a}))$, and so $M \models \psi(\pi(\bar{m}_0), f(\bar{a}))$ i.e. $M \models \psi(f(\bar{m}), f(\bar{a}))$ i.e. $M \models \psi(\hat{m}, f(\bar{a}))$.

Also notice that we ensured at the start that $D\bar{m} \leq M'$. Thus our induction goes through. \square

3.4. Uncountable canonical models

We first note that we have a ready made closure operation which satisfies the conditions to be a pre-geometry (see [5]).

Definition 3.4.1. Given $M \models T$ and any $A \subseteq M$ let the closure operation cl_M be given by:

$$c \in \text{cl}_M(A) \Leftrightarrow d_M(c/A) \leq 0$$

Proposition 3.4.2. For any $M \models T$ we have that cl_M gives a pre-geometry, i.e. it satisfies the following:

1. $\text{cl}_M(A) = \bigcup \{\text{cl}_M(A') \mid A' \subseteq_{\text{fin}} A\}$;
2. $A \subseteq \text{cl}_M(A)$;
3. $\text{cl}_M(\text{cl}_M(A)) = \text{cl}_M(A)$;
4. If $a \in \text{cl}_M(Ab)$ and $a \notin \text{cl}_M(A)$ then $b \in \text{cl}_M(Aa)$;
5. If $X \subseteq \text{cl}_M(Y)$ then $\text{cl}_M(X) \subseteq \text{cl}_M(Y)$;
6. $\text{cl}_M(A) \leq M$.

Proof. See [5]. \square

Definition 3.4.3. We say that $C \subseteq M$ is d_M -**independent** over A if we have $d_M(C'/A) = |C'|$ for every finite $C' \subseteq C$.

We say that $A' \subseteq A$ is a d_M -**basis** of A iff A' is a maximal d -independent subset.

For any definable or type-definable relation S defined over a set of parameters A from M :

$$\dim(S(M^n)) = \max\{d(\bar{s}/A) \mid \bar{s} \in S(M^n)\}.$$

If $S(M^n) = \emptyset$ then let $\dim(S(M^n)) = -1$.

We note that our pre-dimension, δ , and dimension, d , so far introduced have been defined on finite sets of elements, i.e. tuples, not sets of them.

Proposition 3.4.4. For any $M \models T$ and $S(M^n)$ defined over $A \subseteq_{\text{fin}} M$ we have $\dim(S) = 0$ iff $S(M^n) \subseteq (\text{cl}_M(A))^n$.

Proof. We may assume $A \leq M$, expanding A if necessary, so that for any $\bar{c} = \langle c_1, \dots, c_n \rangle \in S(M^n)$ we have $d_M(\bar{c}/A) = d_M(A\bar{c}) - d_M(A) = 0$, i.e. $d_M(A\bar{c}) = d_M(A)$. Thus $d_M(Ac_i) = d_M(A)$ so that $c_i \in \text{cl}_M(A)$ and thus $\bar{c} \in \text{cl}_M(A)^n$. Conversely if $S(M^n) \subseteq (\text{cl}_M(A))^n$ then any $\bar{c} \in S(M^n)$ must be such that $d(c_i/A) = 0$ and so $d(\bar{c}/A) = 0$. \square

Definition 3.4.5. • We say that a model M of T has the **countable closure property** (or **CCP**) if for any finite $A \subseteq M$ we have that $\text{cl}(A)$ is countable.

- A model M of T will be called **canonical** if M has a d -basis B such that M is prime over B .

Lemma 3.4.6. Any d -basis, B , in a model M of T with the countable closure property is of cardinality $= |M|$ and such that $\text{cl}_M(B) = M$.

Proof. By definitions. \square

Theorem 3.4.7. 1. There are canonical models of T in every infinite cardinality;
2. Any canonical model has CCP;
3. Every model with CCP is prime over any d -basis, and so is canonical.

Proof. We first note that by Shelah's Theorem (i.e. 5.17 in [10] or 5.5.1 in [1]), since T is complete, countable and ω -stable, we have that there is a prime model of T over any set of parameters, so 1. is clear.

For 2. let $B = \{b_1, \dots, b_\lambda, \dots\}$ be a set of d -independent elements of cardinality $\kappa \geq 2^{\aleph_0}$, which can be found in a monster model of T , and let $M(B)$ be the prime model over B . Define $B_\lambda = \{b_\alpha \mid \alpha < \lambda\}$, the set of the first λ elements of B , and $M(B_\lambda)$ to be the prime model of T over B_λ .

Note that $M(B_0) = M(\emptyset)$ is infinite, by Axiom 2, but contains no sets of dimension greater than zero, since it contains no d -independent elements.

Now, for any $A_0 \subseteq_{\text{fin}} M(B)$ there is a finite subset $B' \subseteq B$ such that $\text{cl}_{M(B)}(A_0) \subseteq \text{cl}_{M(B)}(B')$. This is by the finite character of closure (3.4.2 1.) and noting that $\text{cl}_{M(B)}(B) = M(B)$ since if $x \notin \text{cl}_{M(B)}(B)$ then $d_{M(B)}(x/B) \geq 1$ which means that $\text{tp}_{M(B)}(x/B)$ is not isolated (it is given by $\{x \neq b\}_{b \in B} \cup \{d_{M(B)}(x) = \delta(x) = 1\}$ by 3.3.3) and so there is a model of T containing B which omits it.

So, taking some zero dimensional $S(M(B)^n)$ defined over A_0 , we have

$$S(M(B)^n) \subseteq \text{cl}_{M(B)}(A_0) \subseteq \text{cl}_{M(B)}(B'),$$

which we will now show is countable. As B' is finite we only need worry about $a \notin B'$, so we claim the following.

Claim 1. For $a \in \text{cl}_{M(B)}(B') \setminus B'$ we have that $\{a' \in M(B) \mid a' \models \text{tp}(a/B')\}$ is in a single orbit over B .

In fact we take a tuple of minimal length $\bar{a} \in M(B)$ extending a such that $d(a/B') = \delta(\bar{a}/B') = d(\bar{a}/B')$ and show that the set $\{\bar{a}' \in M(B) \mid \bar{a}' \models \text{tp}(\bar{a}/B')\}$ is in a single orbit over B , which will clearly do.

Note that since $d(\bar{a}/B') = d(a/B') \leq 0$ and by the d -independence of B we get $0 \leq d(\bar{a}/B) \leq d(\bar{a}/B') \leq 0$ and so we get equality throughout and $d(\bar{a}/B) = \delta(\bar{a}/B)$. Thus we get by Corollary 3.3.3 that $\text{tp}(\bar{a}/B)$ is determined by $\text{qftp}(\bar{a}/B)$. I want to show that it is determined by $\text{tp}(\bar{a}/B')$

We have that \bar{a} is disjoint from B . If it were not we would have $a_i \in B$ for some $a_i \in \bar{a}$. If $a_i \in B \setminus B'$ then $d(B'\bar{a}) > d(B')$ since $B \setminus B'$ is d -independent over B' , contradicting $d(\bar{a}/B') = 0$. If $a_i \in B'$ then we get that $\bar{a}' = \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k \rangle$ is such that $d(B'\bar{a}) = d(B'\bar{a}')$, contradicting the minimality of \bar{a} .

We also have that there cannot be any H-relations between co-ordinates of \bar{a} and elements of $B \setminus B'$ (or involving elements of \bar{a} , $B \setminus B'$, and B') since if there were with b_α and b_β in $B \setminus B'$ say, we would have $r(\bar{a} \dot{\leftrightarrow} B' b_\alpha b_\beta) = r(\bar{a} \dot{\leftrightarrow} B') + k$, for some $k \geq 1$. Noticing also that $r(B') = r(B' b_\alpha b_\beta) = 0$, since B is d -independent, we get

$$\begin{aligned} \delta(\bar{a} B' b_\alpha b_\beta) &= |\bar{a}| + |B' b_\alpha b_\beta| - r(\bar{a}) - r(B' b_\alpha b_\beta) - r(\bar{a} \dot{\leftrightarrow} B_0 b_\alpha b_\beta) \\ &= |\bar{a}| + |B'| + |\{b_\alpha b_\beta\}| - r(\bar{a}) - r(B') - r(\bar{a} \dot{\leftrightarrow} B') - k \\ &= \delta(\bar{a} B') + |\{b_\alpha b_\beta\}| - k. \end{aligned}$$

But this gives $\delta(b_\alpha b_\beta / B' \bar{a}) = |\{b_\alpha b_\beta\}| - k$, contradicting the d -independence of B which, since $\bar{a} \in \text{cl}(B')$, gives us that $\delta(b_\alpha b_\beta / B' \bar{a}) = \delta(b_\alpha b_\beta / B') = |\{b_\alpha b_\beta\}|$.

Thus there are no positive relations at all in $\text{qftp}(\bar{a} / (B \setminus B'))$, and so $\text{qftp}(\bar{a} / B')$ implies the rest of $\text{qftp}(\bar{a} / B)$. Taking this together with the above we have that $\text{tp}(\bar{a} / B)$ is completely determined by $\text{qftp}(\bar{a} / B')$, and so by $\text{tp}(\bar{a} / B')$. Hence the type of the elements in $\{\bar{a}' \in M(B) \mid \bar{a}' \models \text{tp}(\bar{a} / B')\}$ is constant, and so the set is in a single orbit over B , and the claim is proved.

Claim 2. For $a \in \text{cl}_{M(B)}(B')$ we have that $\{a' \in M(B) \mid a' \models \text{tp}(a / B')\}$ is indiscernible over B .

We let $\bar{a} \ni a$ be minimal such that $d(a / B') = \delta(\bar{a} / B')$, and show that $A = \{\bar{a}' \in M(B) \mid \bar{a}' \models \text{tp}(\bar{a} / B')\}$ is indiscernible over B . Enumerate A as $\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_\lambda, \dots\}$. It suffices to show that, for any $n \in \mathbb{N}$ and any $\bar{a}^* = \langle \bar{a}_{i_1}, \dots, \bar{a}_{i_n} \rangle \in A^n$, $\text{tp}(\bar{a}^* / B)$ is decided by n and $\text{tp}(\bar{a} / B')$. We do this by showing that $\text{tp}(\bar{a}^* / B)$ is decided by $\text{qftp}(\bar{a}^* / B)$, that the co-ordinates of \bar{a}^* are disjoint and no H-relations hold involving distinct co-ordinates (and possibly elements of B), so that $\text{qftp}(\bar{a}^* / B)$ is given by $\text{qftp}(\bar{a} / B)$ and n , and then Claim 1 gives us that this is decided by $\text{tp}(\bar{a} / B')$.

To start we show that $B \bar{a}^* \leq M(B)$ so that $\text{tp}(\bar{a}^* / B)$ is decided by $\text{qftp}(\bar{a}^* / B)$, by Corollary 3.3.3. Without loss let $i_j = j$ for $j = 1, \dots, n$. For each i we have that $d(\bar{a}_i / B) \geq 0$ since $B \leq M(B)$. But also we have $d(\bar{a}_i / B') \leq 0$ and $d(\bar{a}_i / B') = \delta(\bar{a}_i / B')$ since $\bar{a}' \models \text{tp}(\bar{a} / B')$ and both those facts are given by the type of \bar{a} over B' . Thus

$$0 \leq d(\bar{a}_i / B) \leq \delta(\bar{a}_i / B) \leq \delta(\bar{a}_i / B') = d(\bar{a}_i / B') \leq 0,$$

and so we get equality for each i . Now, by induction on the formula given in Proposition 3.1.3 we have

$$\delta(\bar{a}_1 \dots \bar{a}_n / B) = \sum_{i=1}^n \delta(\bar{a}_i / B \bar{a}_1 \dots \bar{a}_{i-1}) \leq \sum_{i=1}^n \delta(\bar{a}_i / B) = 0.$$

Again since $B \leq M(B)$ we get $\delta(\bar{a}_1 \dots \bar{a}_n / B) \geq 0$ and so $\delta(\bar{a}_1 \dots \bar{a}_n / B) = 0 = d(\bar{a}_1 \dots \bar{a}_n / B)$ giving us $B \bar{a}_1 \dots \bar{a}_n \leq M(B)$ as required.

Next I show that the \bar{a}_i 's are disjoint. Note that for $i \neq j$

$$|B' \bar{a}_i \bar{a}_j| = |B' \bar{a}_i| + |B' \bar{a}_j| - |B'| - |\bar{a}_i \cap \bar{a}_j|,$$

and

$$r(B' \bar{a}_i \bar{a}_j) = r(B' \bar{a}_i) + r(B' \bar{a}_j) - r(B') - r(B' \dot{\leftrightarrow} \bar{a}_j) + r(B' \bar{a}_i \dot{\leftrightarrow} \bar{a}_j) - r(\bar{a}_i \cap \bar{a}_j).$$

Further note that $r(B'\bar{a}_i \dot{\leftrightarrow} \bar{a}_j) - r(B' \dot{\leftrightarrow} \bar{a}_j) = k \geq 0$, and that, if $\bar{a}_i \cap \bar{a}_j$ is non-empty, then $r(\bar{a}_i \cap \bar{a}_j) - |\bar{a}_i \cap \bar{a}_j| = l > 0$. The second (strict) inequality is because if not then $\delta(\bar{a}_i \cap \bar{a}_j/B') \leq 0$, which contradicts the fact that \bar{a} was minimal such that $\delta(\bar{a}/B') = 0$ (and $\bar{a}_i \models \text{tp}(\bar{a}/B')$). Then, as $B' \leq M(B)$,

$$\begin{aligned} 0 &\leq \delta(\bar{a}_i \bar{a}_j/B') = |B'\bar{a}_i \bar{a}_j| - r(B'\bar{a}_i \bar{a}_j) - |B'| + r(B') \\ &= |B'\bar{a}_i| + |B'\bar{a}_j| - |B'| - |\bar{a}_i \cap \bar{a}_j| - \\ &\quad r(B'\bar{a}_i) + r(B'\bar{a}_j) - r(B') - r(B' \dot{\leftrightarrow} \bar{a}_j) + r(B'\bar{a}_i \dot{\leftrightarrow} \bar{a}_j) - r(\bar{a}_i \cap \bar{a}_j) \\ &= \delta(B'\bar{a}_i) + \delta(B'\bar{a}_j) - 2\delta(B') - \\ &\quad (r(B'\bar{a}_i \dot{\leftrightarrow} \bar{a}_j) - r(B' \dot{\leftrightarrow} \bar{a}_j)) - (r(\bar{a}_i \cap \bar{a}_j) - |\bar{a}_i \cap \bar{a}_j|) \\ &= \delta(\bar{a}_i/B') + \delta(\bar{a}_j/B') - k - l = 0 + 0 - k - l < 0 \end{aligned}$$

and this contradiction proves that the \bar{a}_i 's are disjoint.

To complete the proof of the claim we need to show that there can be no H-relations holding which involve distinct co-ordinates of \bar{a}^* , and possibly elements of B . Such an H-relation can involve at most three of the co-ordinates, without loss \bar{a}_1, \bar{a}_2 and \bar{a}_3 , and the elements $\bar{b} \subseteq_{fin} B$, and would give us that

$$\begin{aligned} r(\bar{b}\bar{a}_1\bar{a}_2\bar{a}_3) - r(\bar{b}) &> (r(\bar{b}\bar{a}_1) - r(\bar{b})) + (r(\bar{b}\bar{a}_2) - r(\bar{b})) \\ &\quad + (r(\bar{b}\bar{a}_3) - r(\bar{b})) \\ \text{i.e. } r(\bar{b}\bar{a}_1\bar{a}_2\bar{a}_3) &> r(\bar{b}\bar{a}_1) + r(\bar{b}\bar{a}_2) + r(\bar{b}\bar{a}_3) - 2r(\bar{b}). \end{aligned}$$

But then we get, replacing \bar{b} with $\bar{b} \cup B'$ if necessary,

$$\begin{aligned} 0 &\leq \delta(\bar{a}_1\bar{a}_2\bar{a}_3/\bar{b}) = |\bar{b}\bar{a}_1\bar{a}_2\bar{a}_3| - r(\bar{b}\bar{a}_1\bar{a}_2\bar{a}_3) - |\bar{b}| + r(\bar{b}) \\ &< |\bar{b}\bar{a}_1| + |\bar{b}\bar{a}_2| + |\bar{b}\bar{a}_3| - 2|\bar{b}| - \\ &\quad (r(\bar{b}\bar{a}_1) + r(\bar{b}\bar{a}_2) + r(\bar{b}\bar{a}_3) - 2r(\bar{b})) - |\bar{b}| + r(\bar{b}) \\ &= \delta(\bar{b}\bar{a}_1) + \delta(\bar{b}\bar{a}_2) + \delta(\bar{b}\bar{a}_3) = 0. \end{aligned}$$

This contradiction shows that there can be no H-relations holding which involve distinct co-ordinates of \bar{a}^* , and possibly elements of B . So we have that $\text{qftp}(\bar{a}^*/B)$ is given by $\text{qftp}(\bar{a}/B)$ and n .

To recap, we have that $\text{tp}(\bar{a}^*/B)$ is decided by $\text{qftp}(\bar{a}/B)$ and n , and from Claim 1 this is decided by $\text{tp}(\bar{a}/B')$. Thus $\{\bar{a}' \in M(B) \mid \bar{a}' \models \text{tp}(\bar{a}/B')\}$ is indiscernible over B .

Claim 3. For $a \in \text{cl}_{M(B)}(B')$ we have that $\{a' \in M(B) \mid a' \models \text{tp}(a/B')\}$ is countable.

This is immediate from the above and Theorem 5.19 in [10] (or Theorem 5.5.1(iii) in [1]) which states that there is no uncountable set of indiscernibles over B in a model prime over B .

Now, any $a \in \text{cl}_{M(B)}(B')$ realises a type over B' , and since there are only countably many types over B' , by ω -stability, and we have just shown that each type $\text{tp}(a/B')$ with $a \in \text{cl}_{M(B)}(B')$ is realised only countably many times we have that $\text{cl}_{M(B)}(B')$ is countable. Thus we have 2.

For 3. Let $|B|$ be a maximal d -independent set in M , so that $|B| = |M|$ and $M = \text{cl}_M(B)$. I show that the type of every element of M is isolated over B . This means that M is atomic over B , and since it has the countable closure property there are clearly no uncountable sets of indiscernibles over B . Then we have from Theorem 5.5.1 in [1] that M is prime over B .

Clearly any $b \in B$ has an isolated type, so consider $m \in M \setminus B$. We have $d(m/B) = 0$ (since $M \subseteq \text{cl}(B)$) so there is $B' \subseteq_{\text{fin}} B$ and a finite tuple \bar{m} containing m such that $\delta(\bar{m}/B') = 0$. From this we get $\delta(\bar{m}/B') = d(\bar{m}/B')$ and so $B'\bar{m} \leq M$. We note that there cannot be any relations between \bar{m} and $B \setminus B'$ since they would contradict B 's independence, so $\text{qftp}(\bar{m}/B')$ decides $\text{qftp}(\bar{m}/B)$. Now by Corollary 3.3.3 we have that the quantifier free type $\text{qftp}(\bar{m}/B')$ decides the complete type $\text{tp}(\bar{m}/B)$. Further, the quantifier free type $\text{qftp}(\bar{m}/B')$ is finite since there are only finitely many possible H-relations and equalities between \bar{m} and B' . So let $\theta(\bar{x}) = \bigwedge \text{qftp}(\bar{m}/B')$. Finally, for any $\varphi(x) \in \text{tp}(m/B)$ there is $\varphi^*(\bar{x}) \in \text{tp}(\bar{m}/B)$ such that $T \vdash \exists x_2, \dots, x_n \varphi^*(\bar{x}) \Rightarrow \varphi(x)$, so for any such $\varphi(x)$ we have that $T \vdash \exists x_2, \dots, x_n \theta(\bar{x}) \Rightarrow \varphi(x)$. Thus M is atomic and so we are done. \square

Theorem 3.4.8. *The class of models of T with the countable closure property is categorical in each uncountable cardinality.*

Proof. Given any two models of T with the countable closure property M and N of the same uncountable cardinality κ we have from the above that they both contain d -independent sets of cardinality κ , C and D respectively. We can take these to be maximal, and there is clearly a partial isomorphism $f : C \rightarrow D$. Also, by the above, N is prime over $D = f(C)$, and M is prime over C . By Theorem 5.5.1 in [1] N and M are isomorphic. \square

From now on we fix a prime model, M of T , over an uncountable d -independent set.

Note 3.4.9. The dimension of the ambient space is n for subsets of M^n since we have a d -independent set of n elements for every $n \in \mathbb{N}$.

By considering realisations of S in an elementary extension of M we see that, in fact, the definition is independent of the parameters over which S is defined since:

$$\begin{aligned} \dim(S(M)) &= \max\{d(\bar{s}/E) \mid \bar{s} \in S(M)\} \\ &= \max\{d(\bar{s}/E) \mid \bar{s} \in S(M') \text{ for } M' \succeq M\} \\ &= \max\{d(\bar{s}/M) \mid \bar{s} \in S(M') \text{ for } M' \succeq M\}. \end{aligned}$$

Also we have for a relation S quantifier free definable over a finite $A \leq M$ that there is $\bar{a} \in S(M^n)$ such that $d(\bar{a}/A) = \delta(\bar{a}/A)$. Thus we can replace the d by δ in the definition of \dim .

Corollary 3.4.10. *This gives us [C1] and [C2].*

4. A new language

We now look for a basis of closed sets for a topology for the model M . By atomic compactness we could define a basis of closed sets to be any sets defined by positive formula in \mathcal{L} and get all the **[Language]** axioms for free. But if we also define the dimension function as above, which is the only way that makes sense here, we immediately come across very simply defined closed sets (which we would want to be analytic) which are irreducible but contain a positively defined subset of the same dimension. This contradicts axiom [C5] of an analytic Zariski structure, and so we search for a more precise language to give us a finer topology.

Example 4.0.11. $S(M^3) = \{\langle x, y, z \rangle \in M^3 \mid R(x, y, z) \& R(y, x, z)\}$ is not expressible as the union of two other \mathcal{L} -positively definable sets but does contain a positively definable subset of the same dimension $\dim(S(M^3)) = 1 = \dim(\{\langle x, y, z \rangle \in S(M^3) \mid x = y\})$.

We embark on what seems to be a more complex route by defining what we call the basic closed sets of M , a new language from which the other definable closed sets of our ‘topology’ will be formed by positive logical operations. I put topology in quotes as infinite intersections of definable closed sets may not be definable, so the definable closed sets do not form a topology. They do however form the closed basis for a topology which we can also define using the language of specialisations introduced in [8]. The topology thus formed will not, however, be compact (as it would have been by atomic compactness if we had just taken all positively defined sets to be closed). By analogy to complex analytic spaces we ‘compactify’ the structure M by adding a new ‘point at infinity’, which satisfies the Hrushovski triple relation with every other pair of elements. The basic closed sets extend to $M \cup \infty = \bar{M}$ in a natural way, where the topology they form proves to be compact.

4.1. Simple closed and special closed sets

Definition 4.1.1. • A relation S , definable over a finite set of parameters, A , on the variables $\bar{x} = \langle x_1, \dots, x_n \rangle$ is called **simple** if it is a finite conjunction of relations of the form $R(x_{i_1}, x_{i_2}, x_{i_3})$ or $x_{j_1} = x_{j_2}$ where the x_{i_k} ’s and x_{j_k} ’s are from \bar{x} or A , i.e. S is a simple relation iff it is positively quantifier free definable in $\mathcal{L} = \{R, =\}$ without using the disjunction symbol.

If S is a simple relation then we call the set $S(M^n) \subseteq M^n$ **simple closed** in M^n .

- An **equational ideal**, $F(\bar{x})$, is a set of equalities between the elements of the tuple of variables \bar{x} and a finite set of parameters which is closed under logical implication.
- We say that a set $\{\phi_i(\bar{x}_i) \mid i \in I\}$ of relations is T -independent (or just **independent**) iff, for any $i \in I$,

$$T \not\models \bigwedge_{j \neq i} \phi_j(\bar{x}_j) \rightarrow \phi_i(\bar{x}_i).$$

- Given a simple closed set $S(M^n)$ and equational ideal $F(M^n)$ over \bar{a} we introduce the notation

$$S_F(M^n) := \{\bar{x} \in S(M^n) \mid M \models F(\bar{x})\};$$

Notation 4.1.2. • Given an equational ideal, I over a finite set, A , we introduce the notation $|I|$ for the size of a maximal set of independent equations in the ideal.

- For a tuple \bar{m} of elements of $M' \succeq M$ and a set $A \subseteq M$ we denote by $htp(\bar{m}/A)$ the subset of $tp(\bar{m}/A)$ of all formulas of the form $R(x_{i_1}, x_{i_2}, x_{i_3})$.
- If S is a simple closed relation over A on the variables \bar{x} and \bar{x}' is a sub-tuple of \bar{x} then we let $htp(\bar{x}'/S)$ be the set of all relations $R(x_{i_1}, x_{i_2}, x_{i_3})$ such that each x_{i_k} is either in \bar{x}' or in A , at least one is in \bar{x}' , and $S(\bar{x}) \vdash R(x_{i_1}, x_{i_2}, x_{i_3})$. We should note here that this notion also makes sense for any type-definable relation S without disjunctions.
- Let $|htp(\bar{x}'/S)|$ be the number of independent H-relations in $htp(\bar{x}'/S)$ modulo the equations in S .

The relations $R(x_1, x_2, x_3)$ and $R(y_1, y_2, y_3)$ are independent whenever $x_1 \neq y_1$, $x_2 \neq y_2$, or $x_3 \neq y_3$.

- If $R(x_{i_1}, x_{i_2}, x_{i_3}) \in htp(\bar{x}'/S)$ we say that S **forces** this relation on \bar{x}' .

Now we define the sets which will be the irreducible sets of our topology. We do this, essentially, by taking the simple closed sets and ‘cutting out’ all the proper simple closed subsets of the same dimension. This is done by recognising that a proper simple closed subset of the same dimension must be defined by extra equalities between co-ordinates (and perhaps parameters) which make some of the H-relations identical. So we just cut-out the parts of a simple closed set which are contained in such equalities (i.e. we remove all proper closed subsets of equal dimension). To do this we first demand that no extra equalities at all hold on any tuples, and secondly allow only the equalities which do cause the dimension of the set to drop (by defining the ‘boundary of the main part’).

Definition 4.1.3. We say that a set S is **free** if we have that:

$$S \not\subseteq \left\{ \bar{x} \in M^n \left| \bigvee_{1 \leq i < j \leq n} x_i = x_j \vee \bigvee_{\substack{1 \leq i \leq n \\ 1 \leq j \leq l}} x_i = a_j \right. \right\}$$

for any parameters $a_1 \dots a_l$, i.e. if $S(M^n)$ is not contained in a non-trivial disjunction of equational ideals.

Note 4.1.4. For $S(\bar{x})$ free and simple closed over a finite set of parameters, A , and any \bar{m} satisfying only the relations forced by S we have

$$\delta(\bar{m}/A) = |\bar{m}| - |htp(\bar{x}/A)|.$$

Definition 4.1.5. If S is simple closed we denote by $\mathcal{I}(S)$ the (finite) set of equational ideals defined over the parameters of S which do not contradict S , but are not implied by S .

Definition 4.1.6. If $S(M^n)$ is a simple closed set define the subset $S^0(M^n) \subseteq S(M^n)$ by:

$$S^0(M^n) = \left\{ \bar{x} \in S(M^n) \left| \bigwedge_{I \in \mathcal{I}(S)} \neg I(\bar{x}) \right. \right\}$$

and the subset $S'(M^n) \subseteq S(M^n)$ by:

$$S'(M^n) = \left\{ \bar{x} \in S(M^n) \mid \bigvee_{I \in \mathcal{I}(S)} I(\bar{x}) \right\}$$

i.e. $S^0(M^n)$ is the set $S(M^n)$ with any tuple with equalities between its co-ordinates, or between its co-ordinates and parameters in S , removed. And $S'(M^n)$ consists of exactly the elements of $S(M^n)$ with any such equalities.

Definition 4.1.7. Given a simple closed $S(M^n)$ defined over the finite set A we define the **boundary of the main part of $S(M^n)$** to be:

$$S^b(M^n) = \bigcup \{S_J(M^n) \mid \dim(S_J(M^n)) < \dim(S^0(M^n)), \\ \text{where } J \text{ is a proper equational ideal over } A\}.$$

We define the **main part** of a simple closed $S(M^n)$, written as $\widehat{S}(M^n)$ to be given by:

$$\widehat{S}(M^n) = S^0(M^n) \cup S^b(M^n).$$

Remark 4.1.8. It is clear that $\dim(S^0(M^n)) = \dim(\widehat{S}(M^n))$.

Definition 4.1.9. A definable set in M^n is called **special closed** if it is one of the following two kinds:

1. a zero dimensional set which is quantifier free definable over any finite parameter set;
2. the main part of a simple closed set, $\widehat{S}(M^n, \bar{a})$, such that

$$\dim(\text{pr}(\widehat{S}(M^n, \bar{a}))) \geq 1$$

for every projection $\text{pr} : M^n \rightarrow M^m$ such that $|\text{pr}(\widehat{S}(M^n, \bar{a}))| > 1$.

Definition 4.1.10. The language \mathcal{L}^* , consists of $=$ and one relation symbol, $\{\widehat{S}_i(-)\}_{i \in \mathbb{N}}$, for each special closed set.

The number of these symbols depends on the number of parameters available in M , and so \mathcal{L}^* has the same cardinality as the structure we are working in.

This language is definably equivalent to $\mathcal{L}(M)$, since basic closed sets are definable in $\mathcal{L}(M)$, and the relation R is basic closed of type 2.

Note 4.1.11. In this section we keep using the notation \widehat{S}_i for symbols of the language \mathcal{L}^* to avoid confusion with elements S of the original language \mathcal{L} .

Example 4.1.12. We do not have atomic compactness for M in \mathcal{L}^* . Since any zero dimensional set is closed we can consider the collection of sets $\{x \mid R(xxx) \& x \neq a\}$ as a varies in $A = \{x \mid R(xxx)\}$. Since A is infinite this set has the finite intersection property, but clearly it has empty intersection. Thus M is not atomic compact in this language. We later introduce a point ∞ which we add to M to form a one point compactification \bar{M} .

Lemma 4.1.13. Given special closed $\widehat{S}(M^n)$ of dimension > 0 the set $\widehat{S'}(M^{n+m}) = \{(\bar{x}, \bar{y}) \in M^{n+m} \mid \widehat{S}(\bar{x})\} = \widehat{S}(M^n) \times M^m$ is also special closed.

Proof. All projections of this set are ≥ 1 dimensional or are singletons (since $\widehat{S}(M^n)$ is special closed), and any equality involving any of the y -variables in $\widehat{S}'(\bar{x}, \bar{y})$ will reduce the dimension, so that the boundaries are still correct (e.g. for any equational ideal I we have that $\dim(S_I \times M^m) < \dim(S^0 \times M^m)$ iff $\dim(S'_I) < \dim(S'^0)$.) \square

Lemma 4.1.14. *For any simple closed $S(M^n)$ there are special closed $U(M^l)$ and $\widehat{T}(M^n)$ of dimension 0 and > 0 respectively such that*

$$\widehat{S}(M^n) = (U(M^l) \times M^{n-l}) \cap \widehat{T}(M^n),$$

up to permutation of co-ordinates.

Proof. If $\dim(\text{pr}(\widehat{S}(M^n))) \geq 1$ for every projection such that $|\text{pr}(\widehat{S}(M^n))| > 1$ then $\widehat{S}(M^n)$ is special closed, and we are done by taking $l = 0$ and $T = S$. So assume $\dim(\text{pr}(\widehat{S}(M^n))) = 0$ for some projection such that $|\text{pr}(\widehat{S}(M^n))| > 1$, so that $\widehat{S}((M^n, \bar{a}))$ is not special closed. Take the projection $\text{pr}_1 : M^n \rightarrow M^l$ with this property and with l maximal. Consider $U = \text{pr}_1(\widehat{S}((M^n)))$. If $l = n$ then $U = \widehat{S}((M^n))$ and is quantifier free definable, zero dimensional, and so special closed and we are done. So assume $l < n$, so that we must have $\dim(\widehat{S}(M^n)) \geq 1$, and can assume that $\text{pr}_1(x_1, \dots, x_n) = (x_1, \dots, x_l)$, due to the proviso ‘up to permutation of co-ordinates’.

We wish to show, firstly, that $U(\bar{x})$ is quantifier free definable, and so special closed, and, secondly, that the ‘remainder’ of the definition of S forms a special closed relation, the meaning of ‘remainder’ to be made clear later.

I claim that $|\text{htp}(\langle x_1, \dots, x_l \rangle / S(M^n))| = l$ i.e. that there are l independent H-relations forced by the definition of S on x_1, \dots, x_l and parameters, all involving at least one x_i . This will mean that $U(\bar{x})$ must be quantifier free, since any H-relations forced between \bar{x} and quantified variables \bar{y} (and possibly parameters \bar{a}) would either

1. force $\delta(\bar{x}\bar{a}\bar{y}) < 0$ and so contradict [Axiom 1](#); or
2. not force $\delta(\bar{x}\bar{a}\bar{y}) < 0$, in which case (since any \bar{x} satisfying U must have $\delta(\bar{x}\bar{a}) = 0$) the existentially quantified variable \bar{y} will be witnessed automatically by [Axiom 2](#) and thus redundant. (i.e. for any $\bar{x} \in U$ we have $\bar{x}\bar{a} \leq \bar{x}\bar{a}\bar{y}_0$ for some \bar{y}_0 witnessing the existentially quantified \bar{y} . By [Axiom 2](#) there is automatically some such \bar{y}_0 .)

We certainly have $|\text{htp}(\langle x_1, \dots, x_l \rangle / S)| \leq l$ or S^0 would be empty by [Axiom 1](#). But say $|\text{htp}(\langle x_1, \dots, x_l \rangle / S)| = l - p$, $p \geq 1$. Then, since $\dim(\text{pr}_1(S^0(M^n))) = 0$ (because $\text{pr}_1(S^0(M^n)) \subseteq \text{pr}_1(\widehat{S}(M^n))$), we would need (for some $q \geq 1$) to have $p + q$ independent H-relations forced by S between (without loss of generality) x_l, \dots, x_{l+q} and x_1, \dots, x_l (and parameters) in order to force the dimension of $\text{pr}_1(S^0(M^n))$ down to zero (i.e. in order to have $|\text{htp}(x_1, \dots, x_{l+q} / S^0(M^n))| = l + q$.) But this contradicts the maximality of l since then there are $(l - p) + (p + q) = l + q$ H-relations forced by $S(M^n)$ between the co-ordinates $x_1, \dots, x_l, \dots, x_{l+q}$, making this projection ‘larger’ and also zero dimensional.

We now remove these l H-relations from the definition of S to get $S_1(M^n)$, simple closed:

$$S_1(M^n) = \{\bar{m} \in M^n \mid \bar{m} \text{ satisfies } S(\bar{x}) \setminus \text{htp}(x_1, \dots, x_l / S(M^n))\}.$$

Note that $\dim(pr_1(S_1^0(M^n))) = l$ since we have just removed l independent H-relations which were forced on the elements of the zero dimensional $pr_1(S^0(M^n))$ to get it ($\dim(pr_1(S_1^0(M^n)))$).

Since $pr_1(S_1^0(M^n)) \subseteq M^l$ any projection of $S_1^0(M^n)$ from M^n onto any $p \geq 1$ of the first l co-ordinates has dimension p . So if the dimension of any projection $pr_2 : S_1^0(M^n)$ were zero dimensional it would have to be onto the co-ordinates x_{l+1}, \dots, x_n . But then $pr_2(S^0(M^n))$ would also be zero dimensional, since the relations forced on x_{l+1}, \dots, x_n by $S(M^n)$ and $S_1(M^n)$ are exactly the same. This would contradict the maximality of l (or the assumption of freeness).

So $\dim(pr(S_1^0)) \geq 1$ for all projections and so \widehat{S}_1 is special closed. We have

$$\widehat{S}(M^n) = (U(M) \times M^{n-l}) \cap \widehat{S}_1(M^n),$$

as required. \square

Proposition 4.1.15. *Any simple closed set $S(M^n)$ in M^n can be written as a finite union of sets of the form:*

$$(U(M^l) \times M^{n-l}) \cap \widehat{T}(M^n) \quad (*)$$

up to permutation of variables, for some $0 \leq k \leq n$, zero dimensional special closed $U(M^l) \subseteq M^l$, and special closed $\widehat{T}(M^n)$ of dimension > 0 .

Proof. We go by induction on the dimension of the ambient space, and so can assume that our simple closed set $S(M^n)$ is free.

If $S(M)$ is simple closed in M^1 , then either $S(M) = M^1$ (i.e. $S(M) = \{x \in M : x = x\}$) in which case $S(M) = \widehat{S}(M) = M^1$ is special closed and of the form required, or $S(M)$ is of dimension zero. The second case occurs if $S(M) = \{x \in M : T(x)\}$ where T is any positive instance of $=$ or R involving x and (possibly) parameters. Then $\dim(S(M)) = 0$ and so $S(M)$ is special closed, and of the form required.

Assuming we have the result for simple closed subsets of M^j for all $j < n$ let $S(M^n)$ be an arbitrary simple closed set in M^n . By freeness we have that $S(M^n) \setminus S^0(M^n) = S'(M^n)$. This is clearly a union of simple closed sets (each S_l is positively defined) and each $S_l(M^n)$ can be considered to be in a lower dimensional space (since each S_l has at least one equality between its co-ordinates). Thus by induction we have the result for each $S_l(M^n)$ and so for the whole of this part of the set.

To complete the proof we show that $S^0(M^n) \subseteq (U(M^l) \times M^{n-l}) \cap \widehat{T}(M^n) \subseteq S(M^n)$ for some special closed $\widehat{T}(M^n)$ and $U(M^l) \subseteq M^l$, \widehat{T} of dimension > 0 and U of dimension 0 . But this is precisely given by 4.1.14, and so our induction goes through. \square

Notation 4.1.16. Given a simple closed $S(M^n)$ and equational ideal I let

$$\mathcal{N}(S, I) = |\text{htp}(\bar{x}/S)| - |\text{htp}(\bar{x}/S_I)|$$

i.e. $\mathcal{N}(S, I)$ is the number of pairs of H-relations in the definition of $S(M^n)$ which become identical under I .

Note 4.1.17. We make the following observations on any free simple closed set S :

1. Considering tuples of distinct elements satisfying only the H-relations stipulated in the definition of S we easily see that, for any equational ideal I :

$$\dim((S_I)^0(M^n)) = \dim(S^0(M^n)) - |I| + \mathcal{N}(S, I).$$

2. The above gives

$$\dim(S_I(M^n)) = \dim(S^0(M^n)) - \min_{J \supseteq I} \{|J| - \mathcal{N}(S, J)\}$$

and so we get that

$$\begin{aligned} S_I(M^n) \subseteq S^b(M^n) & \quad \text{iff} \quad \min_{J \supseteq I} \{|J| - \mathcal{N}(S, J)\} > 0 \\ & \quad \text{iff} \quad \max_{J \supseteq I} \{\mathcal{N}(S, J) - |J|\} < 0. \end{aligned}$$

3. We may get $\dim(S^0) < \dim(S)$ if $\min\{|J| - \mathcal{N}(S, J) \mid J \text{ is an equational ideal}\} < 0$.
4. Also, for S, T simple closed, $\mathcal{N}_{S \cap T, I} \geq \mathcal{N}(S, I)$ since if I makes n pairs of H-relations in the definition of S identical it must make $\geq n$ such pairs identical in the definition of $S \cap T$.
5. Given simple closed S and equational ideals I and J we clearly have that $\mathcal{N}_{S, (I \cup J)} \geq \mathcal{N}_{S_I, J}$ since $\text{htp}(\bar{x}/(S_I)_J) = \text{htp}(\bar{x}/S_{(I \cup J)})$ but $|\text{htp}(\bar{x}/S)| \geq |\text{htp}(\bar{x}/S_I)|$.

Proposition 4.1.18. *Finite intersections of special closed sets of dimension > 0 can be expressed as finite unions of sets of the form (*) from Proposition 4.1.15.*

Proof. Let $\widehat{S}(M^n) = S^0(M^n) \cup S^b(M^n)$ and $\widehat{T}(M^n) = T^0(M^n) \cup T^b(M^n)$ be special closed of dimension > 0 , so that $\dim(\text{pr}(\widehat{S}(M^n))) > 0$ and $\dim(\text{pr}(\widehat{T}(M^n))) > 0$ for all projections. By induction on the dimension of the ambient space we can assume that both S and T are free. We note that here we are only interested in realisations in M , and so for the rest of this proof will leave out M in our notation. Thus we have:

$$\begin{aligned} \widehat{S} \cap \widehat{T} &= (S^0 \cup S^b) \cap (T^0 \cup T^b) \\ &= (S^0 \cap T^0) \cup (S^b \cap T^b) \end{aligned} \tag{1}$$

$$= (S \cap T)^0 \cup (S^b \cap T^b) \tag{2}$$

since (1) $S^0 \cap T^b = T^0 \cap S^b = \emptyset$, because for any $\bar{x} \in S^0$ we have $\bigwedge_{i \neq j} (x_i \neq x_j)$, whereas for any $\bar{x} \in T^b$ we have $x_i = x_j$ for some $i \neq j$, and (2)

$$\begin{aligned} \bar{x} \in S^0 \cap T^0 &\Leftrightarrow (\bar{x} \in S \ \& \ \bigwedge_{i \neq j} (x_i \neq x_j) \ \& \ \bar{x} \in T \ \& \ \bigwedge_{i \neq j} (x_i \neq x_j)) \\ &\Leftrightarrow \bar{x} \in S \ \& \ \bar{x} \in T \ \& \ \bigwedge_{i \neq j} (x_i \neq x_j) \Leftrightarrow \bar{x} \in (S \cap T)^0. \end{aligned}$$

Now, given S, T simple closed, and I an equational ideal, if: $\min_{J \supseteq I} \{|J| - \mathcal{N}(S \cap T, J)\} > 0$ then (since for each $J \supseteq I$ we have $|J| - \mathcal{N}(S, J) \geq |J| - \mathcal{N}(S \cap T, J)$)

we get $\min_{J \supseteq I} \{|J| - \mathcal{N}(S, J)\} > 0$. From this and [Note 4.1.17](#) part 3, given S, T simple closed we get:

$$S^b = \bigcup \{S_I \mid \min_{J \supseteq I} \{|J| - \mathcal{N}(S, J)\} > 0\}$$

and then see that:

$$S^b \cap T^b = (S \cap T)^b \cup U$$

where

$$U = \bigcup \{(S \cap T)_I : \min_{J \supseteq I} (|J| - \mathcal{N}(S, J)) > 0 \ \& \ \min_{J \supseteq I} (|J| - \mathcal{N}(T, J)) > 0 \\ \& \ \min_{J \supseteq I} (|J| - \mathcal{N}(S \cap T, J)) \leq 0\}.$$

This U is positively defined, and so a finite union of simple closed sets, and so by [Proposition 4.1.15](#) can be expressed as a union of sets of the form $(*)$ as required.

From (2) above we get:

$$\widehat{S} \cap \widehat{T} = (S \cap T)^0 \cup (S \cap T)^b \cup U \\ = \widehat{(S \cap T)} \cup U.$$

Now, $S \cap T$ is simple closed, and so by [Lemma 4.1.14](#) it is also a union of sets of the form $(*)$ as required, and we are done. \square

4.2. The point ∞

Definition 4.2.1. Let $\mathcal{L}_\infty^* = \mathcal{L}^* \cup \{\infty\}$ be the language obtained from \mathcal{L}^* by adding the constant symbol ∞ . Then $\bar{M} = M \dot{\cup} \{\infty\}$ is a model describable in \mathcal{L}_∞^* . We write T' for the theory in \mathcal{L}_∞^* obtained from our old theory (T) in \mathcal{L}^* simply by restricting all quantifiers to elements of M (i.e. adding $x_i \neq \infty$ after every quantifier).

Definition 4.2.2. In \bar{M} we interpret each symbol $S \in \mathcal{L}^*$, other than $=$, by

$$S(\bar{M}) = S(M) \cup (\bar{M}^n \setminus M^n).$$

Note 4.2.3. Where the projectivisation of M was denoted by \mathbf{P} (short for $\mathbf{P}(M)$) in the axioms for an analytic Zariski structure in Chapter 1, we use the notation \bar{M} here for convenience.

Definition 4.2.4. A relation is called **basic closed** if it is a symbol of \mathcal{L}^* or an equational ideal defined in it with parameters. The realisation of a basic closed relation is called a basic closed set.

Remark 4.2.5. Given any special closed n -relation $U(-)$ we have that any tuple \bar{m} which contains ∞ and is of length n will satisfy the relation. i.e. $\bar{M} \models U(\bar{m})$ for any special closed relation $U(-)$ and $\bar{m} \ni \infty$.

Notation 4.2.6. By an abuse of notation we write T for the complete theory of \bar{M} translated from $\mathcal{L}(M)$ into the new language \mathcal{L}^* , and T_∞ for the complete theory of \bar{M} translated from $\mathcal{L}_\infty(\bar{M})$ into the $\mathcal{L}^* \cup \{\infty\}$.

Notation 4.2.7. We see our old definitions of δ and d must be altered in this structure, or else we would have $d_{\bar{M}}(A)$ unbounded below for any $A \subseteq \bar{M}$, since $r(\infty \cup AX) > 3|AX|$ for any extension of A . Thus we abuse the notation, and for any $A \subseteq_{fin} \bar{M}$ let:

$$\delta(A) = \delta(A \setminus \{\infty\}).$$

With this alteration all the definitions make sense.

Note 4.2.8. We note that, for an elementary extension $M' \geq M$, we have that $\dim(S(M')) = \dim(S(M))$ for any definable S , but that for some S we have $\dim(S(\bar{M})) > \dim(S(M))$.

Example 4.2.9. If $S = \{\langle x, y, z \rangle \mid R(x, y, z) \ \& \ R(y, z, x) \ \& \ R(z, x, y)\}$ then $\dim(S(M)) = 3 - 3 = 0 < 2 = \dim(S(\bar{M}))$, since taking $x = \infty$ we can take any $\langle y_0, z_0 \rangle$, and get $\langle \infty, y_0, z_0 \rangle \in S(\bar{M})$, so that $\{\infty\} \times \bar{M}^2 \subseteq S(\bar{M})$.

Definition 4.2.10. • A relation S is called \mathcal{L}^* -closed if it is positively definable in \mathcal{L}_∞^* (with parameters). A subset of \bar{M}^n is called \mathcal{L}^* -closed iff it is the realisation of a closed relation in \bar{M}^n .

- A set $S \subseteq \bar{M}^n$ is called **closed** if it is the intersection $\bigcap_{i \in I} C_i$, with C_i \mathcal{L}^* -closed, and I any index set. Subsets of \bar{M}^n of the form $\bar{M}^n \setminus S$ for a closed S are called **open**. We write $U \subseteq_{op} P^n$ to say that U is open and $S \subseteq_{cl} P^n$ to say that S is closed.

We say that a set $O \subseteq U$ is **open in U** iff it is the compliment in U of a closed set in U (write $X \subseteq_{op} U$). If we say simply that X is closed we mean in \bar{M}^n .

- Given an open $U \subseteq \bar{M}^n$ we say that a set $X \subseteq U$ is \mathcal{L}^* -closed in U iff $X = C \cap U$ for some set C , \mathcal{L}^* -closed in \bar{M}^n .

Corollary 4.2.11. This gives us axioms [A1], [A2], [A4], where we take the collection \mathcal{C} to be the collection of all \mathcal{L}^* -closed sets.

Remark 4.2.12. Note that any simple closed $S(M^n)$ is closed, since if it is positively definable in $\mathcal{L}(M)$ then it is positively definable in \mathcal{L}^* by Proposition 4.1.15. But also note that there are more closed sets definable in \mathcal{L}^* than the simple closed sets of $\mathcal{L}(M)$.

Proposition 4.2.13. All \mathcal{L}^* -closed sets (in some open set U) are positive quantifier-free definable (in the open set U). That is, they are definable without the use of the negation symbol or any quantifiers.

Proof. We show that for any quantifier free \mathcal{L}^* -closed set $C(\bar{M}^n)$ in \bar{M}^n we have that $\{\bar{x} \in \bar{M}^{n-1} \mid \bar{M} \models \exists x_n C(\bar{x}, x_n)\}$ and $\{\bar{x} \in \bar{M}^{n-1} \mid \bar{M} \models \forall x_n C(\bar{x}, x_n)\}$ are quantifier free definable. This will clearly do.

First note that we can assume that our set is free, since if $C(\bar{M}^n) \subseteq \{\langle \bar{x}, x_n \rangle \in \bar{M}^n \mid \bigvee_{1 \leq n} I_i(\bar{x}, x_n)\}$ where the $I_i(\bar{x}, x_n)$ are equational ideals, then we can consider each $C(\bar{M}^n) \cap I_i(\bar{x}, x_n)$ separately. Each can be considered to be closed in the ambient space \bar{M}^{n-1} (for some $l > 0$), and by induction to be free there. Our proof shows that each $C(\bar{M}^n) \cap I_i(\bar{x}, x_n)$ is quantifier free definable and then we can reconstruct $C(\bar{M}^n)$ as the union of them.

Now we deal with the universal quantifier. If $C(\bar{x}, x_n)$ forces any equality between x_n and any other (non-quantified) variable or parameter then $\forall x_n C(\bar{x}, x_n)$ is contradictory.

Thus we can assume that $C(\bar{x}, x_n)$ does not force any equality between x_n and any other (non-quantified) variable or parameter.

If $C(\bar{x}, x_n)$ forces any special closed relation between x_n and any other variable (x_i say) then we must have $x_i = \infty$ or we would violate [Axiom 1](#), as ∞ is the only element which can satisfy an H-relation with all other elements. Thus we can replace any H-relations in $\forall x_n C(\bar{x}, x_n)$ which involves x_n and any other x_i , by $x_i = \infty$. Hence we can remove any non-contradictory universal quantifiers from the definition of $\forall x_n C(\bar{x}, x_n)$ and we are done with \forall .

To deal with the existential quantifier let $C(\bar{x}, x_n) = \bigvee_{i=1}^l C_i(\bar{x}, x_n)$ be the quantifier free definition of $C(\bar{M}^n)$, where each C_i is a conjunction of special closed relations. Without loss we can let $l = 1$ since if $\exists x_n C_1(\bar{x}, x_n)$ and $\exists x_n C_2(\bar{x}, x_n)$ are both quantifier free definable then so is $\exists x_n (C_1(\bar{x}, x_n) \vee C_2(\bar{x}, x_n))$. So $C(\bar{x}, x_n)$ is a positive conjunction of special closed relations. Say $C(\bar{x}, x_n) = U_1(\bar{x}^1) \wedge \cdots \wedge U_k(\bar{x}^k)$ where each U_i is special closed and \bar{x}^i is a sub-tuple of (\bar{x}, x_n) .

If $x_n \notin \bar{x}^i$ for all i then clearly $\exists x_n U_i(\bar{x}^i) \equiv U_i(\bar{x}^i)$ for all i , so we get a result.

Otherwise consider all the $U_j(\bar{x}^j)$ with $x_n \in \bar{x}^j$. Define the relation $C'(\bar{x})$ to be obtained from $C(\bar{x}, x_n)$ simply by omitting all $U_j(\bar{x}^j)$ with $x_n \in \bar{x}^j$ from the definition of $C(\bar{x}, x_n)$. I claim that $C'(M^{n-1}) = \{\bar{x} \in \bar{M}^{n-1} \mid \bar{M} \models \exists x_n C(\bar{x}, x_n)\}$.

Consider \bar{m} such that $\bar{M} \models \exists x_n C(\bar{m}, x_n)$. There is $m_n \in \bar{M}$ such that we get $\bar{M} \models C(\bar{m}, m_n)$. Now, by the definition of $C'(\bar{x})$ we have that $\bar{M} \models C'(\bar{m})$. Thus $\{\bar{x} \in \bar{M}^{n-1} \mid \bar{M} \models \exists x_n C(\bar{x}, x_n)\} \subseteq C'(M^{n-1})$.

Consider $\bar{m} \in C'(M^{n-1})$ and let $\bar{m}' = \langle m_1, \dots, m_{n-1}, \infty \rangle$. Now consider the $U_j(\bar{x}^j)$ with $x_n \in \bar{x}^j$ in the definition of $C(\bar{x}, x_n)$ (i.e. the ones we removed to get the definition of $C'(\bar{x})$). We have from [Remark 4.2.5](#) that $\bar{M} \models U_j(\bar{m}^j)$, where \bar{m}^j is the tuple corresponding to \bar{x}^j from \bar{m}' , since this tuple contains ∞ . Since we also have $\bar{M} \models C'(\bar{m})$ we have that $\bar{M} \models C(\bar{m}, \infty)$, and so all special closed relations in C are satisfied and we get $\bar{M} \models \exists x_n C(\bar{m}, x_n)$.

So $C'(M^{n-1}) = \{\bar{x}' \in \bar{M}^{n-1} \mid \bar{M} \models \exists x_n C(\bar{x}', x_n)\}$ and we are done with \exists . \square

Note 4.2.14. We note that this is the first instance that we use the special properties of ∞ to get a result (here we use the fact that any positive existential statement can be witnessed by ∞). This is very unlike the position of ∞ in \mathbb{C} , the projective line, where it is treated like any other element. The difference is due to the fact that we have no algebraic structure in our context, so that there is no mechanism to pull ∞ into the structure. This contrasts with the Hrushovski-type structures presented in [18], or in [12] (which also have analytic-type properties) where we have a field structure, which makes the models much more homogenous.

Corollary 4.2.15. *All closed sets are given by positive quantifier-free types in \mathcal{L}^* , and any positive quantifier-free type gives a closed set.*

Corollary 4.2.16. *Projections of closed sets are closed.*

Proof. Let $C(\bar{x}, x_n)$ be a positive quantifier-free type, giving a closed set, $C(\bar{M}^n)$. Then consider the projection

$$pr(C(\bar{M}^n)) = \{\bar{x} \in \bar{M}^{n-1} \mid \bar{M} \models \exists x_n C(\bar{x}, x_n)\}.$$

We note that we can assume that x_n is not bound (i.e. in a relation $x_n = a$ for some parameter a) as if it were we would simply get $pr(C(\bar{M}^n)) = C(\bar{M}^{n-1}, a) \times \{a\}$, which is quantifier free. Thus the quantifier-free type $C'(\bar{x})$ obtained from $C(\bar{x}, x_n)$ by removing any special closed relation involving an occurrence of x_n really does not involve x_n at all. Using the same argument as in 4.2.13 we see that all the special closed relations in $C(\bar{x}, x_n)$ involving an occurrence of x_n can be realised by ∞ , and thus that $pr(C(\bar{M}^n)) = C'(\bar{M}^{n-1})$. \square

Corollary 4.2.17. *Any \mathcal{L}^* -closed set in M^n is a finite union of sets of the form:*

$$S(M^n) \cap (U(M^l) \times M^{n-l})$$

up to permutation of the variables, where $S(M^n)$ is special closed of dimension ≥ 1 and $U(M^{n-l})$ is zero dimensional special closed and $0 \leq l \leq n$.

Proof. By Propositions 4.2.13 and 4.1.18 we only need to show that $S \times T$ and $S \times U$ are of the form required, for S, T special closed of dimension > 0 and U special closed of dimension 0.

Any part of the definition of a closed set in M defined by equalities can be defined by special closed sets. Thus we do not need to consider equational ideals in M .

We note that for special closed $S(\bar{M}^n)$ and $T(\bar{M}^m)$ of dimension ≥ 1 we have:

$$\begin{aligned} (S(\bar{M}^n) \times T(\bar{M}^m)) \cap M^{n+m} &= S(M^n) \times T(M^m) \\ &= S'(M^{n+m}) \cap T'(M^{n+m}) \end{aligned}$$

where $S'(M^{n+m}) = \{\langle \bar{x}, \bar{y} \rangle \in M^{n+m} \mid S(\bar{x})\} = S(M^n) \times M^m$ and $T'(M^{n+m}) = \{\langle \bar{x}, \bar{y} \rangle \in M^{n+m} \mid T(\bar{y})\} = T(M^m) \times M^n$ are both special closed by Lemma 4.1.13.

Thus, by Proposition 4.1.18, we have that $S \times T$ is of the form required.

Also, given zero dimensional $U \in M^m$ we have:

$$(S(\bar{M}^n) \times (U \cup (\bar{M}^m \setminus M^m))) \cap M^{n+m} = T(M^{n+m}) \cap (M^n \times U)$$

for the set $T(M^{n+m}) = \{\langle \bar{x}, \bar{y} \rangle \in M^{n+m} \mid S(\bar{x})\} = S(M^n) \times M^m$ which is special closed again by Lemma 4.1.13.

The result is then immediate from Propositions 4.2.13 and 4.1.18. \square

Lemma 4.2.18. *For any $C \subseteq_{cl} U \subseteq_{op} \bar{M}^n$ there is a zero dimensional set O (the infinite intersection of zero dimensional special closed sets) and an \mathcal{L}^* -closed set S such that $C \cong (O \times S) \cap U$ where \cong means ‘= up to permutation of co-ordinates’.*

Proof. Let $C = \bigcap_{i \in I} C_i$ for C_i special closed or equational ideals. If infinitely many of the C_i contain distinct H-relations then $C(M^n)$ is empty by Axiom 1. Thus all such an infinite set does is force some variables to be equal to ∞ , and so we can replace any such infinite set of C_i ’s simply by a formula insisting some variables be ∞ . Thus only finitely many of the C_i ’s can involve distinct H-relations. Clearly only finitely many of them can be equational ideals, or involve distinct equalities, and so only finitely many of them are special closed sets of dimension > 0 or equational ideals, since distinct such sets must contain distinct H-relations or equalities. Thus all but finitely many of the C_i are zero dimensional special closed sets with the same H-relations, but different inequalities.

Supposing that all these zero dimensional special closed relations are on the first l co-ordinates we get the result. \square

4.3. Compactness

Definition 4.3.1. Given an elementary extension $\tilde{M} \geq \bar{M}$, a **specialisation** for our closed sets is simply a map, $\pi : \tilde{M} \rightarrow \bar{M}$, such that for any closed set $S(\bar{M}^n)$ we have $\bar{a} \in S(\bar{M}^n) \Rightarrow \pi(\bar{a}) \in S(\bar{M}^n)$.

We note that in particular we have $\pi(a) = a$ for $a \in \bar{M}$ and that this is simply a homomorphism.

Lemma 4.3.2. Suppose $\tilde{M} \geq \bar{M}$ and $\bar{M} \subseteq \bar{D} \subseteq \tilde{M}$ is such that $\bar{D} \leq \tilde{M}$. Then we can extend any partial homomorphism $\pi : \bar{D} \rightarrow \bar{M}$ such that $\pi|_{\bar{M}} = id_{\bar{M}}$ to a full homomorphism $\pi : \tilde{M} \rightarrow \bar{M}$ in a non-trivial way (i.e. so that $\pi(m) \neq \infty$ for some $m \in \tilde{M} \setminus \bar{D}$).

Proof. We proceed by induction, so choose $m \in \tilde{M} \setminus \bar{D}$.

If $d(m/D) = 0$ then we take a minimal length tuple \bar{m} in \tilde{M} extending m such that $\delta(\bar{m}/D) = 0$. Note that we can assume that all the co-ordinates of \bar{m} are from $\tilde{M} \setminus \bar{D}$ and are distinct. We can also assume that there is no $\bar{m}_0 \subsetneq \bar{m}$ such that $\delta(\bar{m}_0/D) = 0$, as if there were then we could consider \bar{m}_0 in place of \bar{m} , and we would still be going forward in the induction.

For any special closed relation $S(\bar{x}, \bar{y})$ from \mathcal{L}^* and $\bar{a} \in \bar{D}^n$ such that $\tilde{M} \models S(\bar{m}, \bar{a})$ we have that $\bar{M} \models S(\infty, \pi(\bar{a}))$ by Remark 4.2.5. So we let $\pi(\bar{m}) = \infty$, and this extends the homomorphism. In fact this is the only possible choice for $\pi(\bar{m})$, though we do not prove this here.

If $d_{M'}(m/D) = 1$ then we already have that $Dm \leq M'$, since then $\delta(mX/A) \geq 1 \forall X \subseteq_{fin} M' \ \& \ \forall A \subseteq_{fin} D$ and clearly $\delta(m/A) \leq 1$ so that for any $X \subseteq_{fin} M'$ and $A \subseteq_{fin} D$ we have $\delta(X/Am) = \delta(mX/A) - \delta(m/A) \geq 1 - 1 = 0$. Thus $\delta(X/Dm) \geq 0$ for any $X \subseteq_{fin} M'$, i.e. $Dm \leq M'$.

Also we have that $r(m) + r(m \leftrightarrow D) = 0$ since clearly $r(m) = 0$ and if $r(m \leftrightarrow D) > 0$ let k_1, k_2 be such that $r(m \leftrightarrow \{k_1, k_2\}) = 1$. Then:

$$\begin{aligned} \delta(m/\{k_1, k_2\}) &= (|\{k_1, k_2\}| + 1 - r(m) - r(\{k_1, k_2\}) - r(m \leftrightarrow \{k_1, k_2\})) \\ &\quad - (|\{k_1, k_2\}| - r(\{k_1, k_2\})) \\ &= 1 - 1 = 0 \end{aligned}$$

which contradicts $d_{M'}(m/D) = 1$.

Since there are no H-relations between m and D there are no special closed relations between them and so we can choose any element, b say, of \bar{M} to be $\pi(m)$, and we still preserve all special closed relations. For let $S(x, \bar{y})$ be a special closed relation from \mathcal{L}^* and \bar{a} a finite tuple of parameters from \bar{D} and suppose $\tilde{M} \models S(m, \bar{a})$. Then S can say nothing about the co-ordinate x , and so $\bar{M} \models S(b, \pi(\bar{a}))$ for any $b \in \bar{M}$, and so we can put $\pi(m) = b$ and this extends the homomorphism.

Corollary 4.3.3. Taking $\bar{D} = \bar{M}$, this lemma gives us atomic compactness for \bar{M} in \mathcal{L}^* .

Theorem 4.3.4. \bar{M} is compact with respect to the closed sets: any family of closed sets with the finite intersection property has non-empty intersection, and the projection of any closed set is closed.

Proof. The theorem follows immediately from the lemma above and the theorems in [13] or [4] mentioned in 3.3.7. \square

Corollary 4.3.5. We have axiom [A3] in the expanded Hrushovski structure \bar{M} .

4.4. The π -topology

Definition 4.4.1. A pair (\tilde{M}, π) where $\tilde{M} \geq \bar{M}$ and $\pi : \tilde{M} \rightarrow \bar{M}$ is a specialisation, such that for any $M' \geq \tilde{M}$, finite $A \subseteq M'$ and specialisation $\pi' : \tilde{M}A \rightarrow \bar{M}$, there is an elementary embedding $f : A \rightarrow \tilde{M}$ such that $\pi' = \pi \circ f$ on A , is called a **universal specialisation**.

Since we have atomic compactness we can use exactly the same construction as in [14] and [11] to produce a universal specialisation.

Note 4.4.2. Throughout this section, as well as fixing our uncountable model \bar{M} of T_∞ , we fix an elementary extension of it \tilde{M} , and a specialisation $\pi : \tilde{M} \rightarrow \bar{M}$, such that (\tilde{M}, π) is a universal specialisation.

Definition 4.4.3. For any $a \in \bar{M}$ we say that the infinitesimal neighbourhood of a is the set:

$$\mathcal{V}_a = \{a' \in \tilde{M} \mid \pi(a') = a\}.$$

Definition 4.4.4. Let $S(\bar{M}^n)$ be a type-definable subset of \bar{M}^n (so that $S(\bar{x})$ is an n -type). We call $S(\bar{M}^n)$ π -closed iff $\pi(S(\tilde{M}^n)) = S(\bar{M}^n)$.

$U \subseteq \bar{M}^n$ is π -open iff it is complement in \bar{M}^n to a π -closed set.

Note 4.4.5. 1. If $S(\bar{M}^n)$ is closed then, since specialisations preserve positive formula, and $S(\bar{x})$ is positive, we have $\pi(S(\tilde{M}^n)) = S(\bar{M}^N)$, so that $S(\bar{M}^n)$ is π -closed also.
2. Hence, if $U(\bar{M}^n)$ is open then $\neg U(\bar{M}^n)$ is closed, and thus π -closed, and so $U(\bar{M}^n)$ is π -open.

Proposition 4.4.6. $C(\bar{M}^n)$ is π -closed iff the type C is given by: $C = \bigwedge_{\lambda \in \Lambda} C_\lambda$ for \mathcal{L}^* -closed relations C_λ .

Remark 4.4.7. If Λ is finite then $C(\bar{M}^n)$ is actually \mathcal{L}^* -closed.

Proof. By 5.1.13 of [2] a set of formula is positive iff it is preserved under homomorphisms. Since (\tilde{M}, π) is universal and \bar{M} is prime any homomorphism $f : X \rightarrow \bar{M}$ (where $X \models T_\infty$) can be seen as a trivial extension of π , and thus $C(\bar{M}^n)$ is preserved under homomorphisms iff it is preserved under π i.e. iff it is π -closed. \square

Corollary 4.4.8. This implies that our closed sets are precisely the same as the π -closed sets.

Remark 4.4.9. Since we have atomic compactness for closed sets, we have compactness for π -closed sets, i.e. the π -topology is compact.

Proposition 4.4.10. *If $\tilde{M}^n \setminus U(\tilde{M}^n)$ is type definable then $U(\tilde{M}^n) \subseteq \tilde{M}^n$ is π -open iff $\bigcup_{a \in U(\tilde{M}^n)} \mathcal{V}_a \subseteq U(\tilde{M}^n)$.*

Proof. For the left to right direction assume that $U(\tilde{M}^n) = \neg C(\tilde{M}^n)$, $C(\tilde{M}^n)$ being π -closed, so that $U(\tilde{M}^n)$ is π -open. Then for any $a \in U(\tilde{M}^n)$ we get:

$$\begin{aligned} \mathcal{V}_a &= \pi^{-1}(a) = \{a' \in \tilde{M} \mid \pi(a') = a\} \\ &\subseteq \{a' \in \tilde{M} \mid \pi(a') \in U(\tilde{M}^n)\} \\ &= \{a' \in \tilde{M} \mid \pi(a') \notin C(\tilde{M}^n)\} \\ &= \{a' \in \tilde{M} \mid a' \notin \pi^{-1}(C(\tilde{M}^n))\} \\ &= \tilde{M}^n \setminus \pi^{-1}(C(\tilde{M}^n)) \\ &= \tilde{M}^n \setminus C(\tilde{M}^n) \\ &= U(\tilde{M}^n) \end{aligned}$$

since $\pi(C(\tilde{M}^n)) = C(\tilde{M}^n)$ because $C(\tilde{M}^n)$ is π -closed. Thus, for any $a \in U(\tilde{M}^n)$, $\mathcal{V}_a \subseteq U(\tilde{M}^n)$, and we are done.

For the other direction we assume the condition on infinitesimal neighbourhoods and show that $\tilde{M}^n \setminus U(\tilde{M}^n)$ is π -closed, so that $U(\tilde{M}^n)$ is π -open. So take $a \in \tilde{M}^n \setminus U(\tilde{M}^n)$. If $\pi(a) \in U(\tilde{M}^n)$ then the condition says that $\mathcal{V}_{\pi(a)} \subseteq U(\tilde{M}^n)$. Clearly $a \in \mathcal{V}_{\pi(a)}$, so this gives $a \in U(\tilde{M}^n)$, contradicting our choice of a . Thus $\pi(a) \notin U(\tilde{M}^n)$, i.e. $a \in \tilde{M}^n \setminus U(\tilde{M}^n)$. Since a was arbitrary in $\tilde{M}^n \setminus U(\tilde{M}^n)$ this means that $\tilde{M}^n \setminus U(\tilde{M}^n)$ is π -closed, and we are done. \square

Proposition 4.4.11 (A5). *Projections are open maps in the π -topology. (i.e. the projection of a π -open set is π -open).*

Proof. We first show that for any $a \in \tilde{M}^n$ and standard projection $pr: \tilde{M}^n \rightarrow \tilde{M}^{n-1}$ we have $pr(\mathcal{V}_a) = \mathcal{V}_{pr(a)}$. Let $a = \langle a_1, \dots, a_{n-1}, a_n \rangle$. That $pr(\mathcal{V}_a) \subseteq \mathcal{V}_{pr(a)}$ is clear since specialisations operate co-ordinate-wise, so that if $\langle a'_1, \dots, a'_{n-1} \rangle \in pr(\mathcal{V}_a)$ then $\exists a'_n$ such that $\langle a'_1, \dots, a'_{n-1}, a'_n \rangle \in \mathcal{V}_a$, and then $\pi(a'_i) = a_i$ for each i , and $pr(a) = \langle a_1, \dots, a_{n-1} \rangle$, so $\langle a'_1, \dots, a'_{n-1} \rangle \in \mathcal{V}_{pr(a)}$. The other direction is also clear since if $\langle a'_1, \dots, a'_{n-1} \rangle \in \mathcal{V}_{pr(a)}$ then $\pi(a'_i) = a_i$ for each i , and clearly $\exists a'_n$ such that $\pi(a'_n) = a_n$ (e.g. a_n itself) and so $\langle a'_1, \dots, a'_{n-1} \rangle \in pr(\mathcal{V}_a)$.

Now let U be π -open in \tilde{M}^n and consider $a \in pr(U)$. Say $\langle a, b \rangle \in U$, and note that, since U is open by Proposition 4.4.10 we have $\mathcal{V}_{\langle a, b \rangle} \subseteq U$. This gives $pr(\mathcal{V}_{\langle a, b \rangle}) \subseteq pr(U)$ which, by the above gives $\mathcal{V}_{pr(\langle a, b \rangle)} = \mathcal{V}_a \subseteq pr(U)$. Again by Proposition 4.4.10 this gives that $pr(U)$ is π -open. \square

Corollary 4.4.12. *We have all the [Language] axioms in the basic Hrushovski structure.*

Proof. The fact that the π -topology coincides exactly with that given by our closed sets means that the above gives this immediately. \square

5. Analytic sets

We show here that the main concepts of an analytic Zariski geometry are realised naturally in the Hrushovski structure, and prove the remaining analytic Zariski axioms.

5.1. Irreducible and strongly irreducible sets

Definition 5.1.1. Let U be open in \bar{M}^n then:

- We say that a set $X \subseteq U$ is **strongly irreducible in U** iff it is relatively \mathcal{L}^* -closed in U , and there is no set X_1 , closed in U , such that $X_1 \subsetneq X$ and $\dim(X) = \dim(X_1)$.
- Given a closed $S \subseteq \bar{M}^n$ and $u \in \bar{M}^n$ we say that S is **analytic at u** iff there is an open $V_u \ni u$ such that $V_u \cap S$ can be expressed as a finite union of sets strongly irreducible in V_u .
- If for any $u \in U$ there is such an open neighbourhood of u , we say that $S \cap U$ is **analytic in U** and write $S \cap U \subseteq_{an} U$.
- If $S \subseteq_{an} U$ then S is **irreducible** iff it is that there are no sets $S_1, S_2 \subseteq_{an} U$ such that $S_i \subsetneq S$ and $S = S_1 \cup S_2$.

If S is closed (in \bar{M}^n) and O is open, we abuse the terminology and say that S is irreducible or analytic in O if $S \cap O$ is.

Note 5.1.2. Whilst we have defined strongly irreducible sets as required, and dimension clearly extends to closed subsets (see [Definitions 2.3.1](#) and [2.3.2](#)) we cannot say that our analytic sets are naturally defined (see [Definition 2.3.3](#)) until we have proved firstly that axiom [C6] holds for all strongly irreducible closed subsets, and secondly that strongly irreducible closed sets and irreducible analytic sets coincide. We are not able to do this until [Section 5.4](#).

Definition 5.1.3. Let $S(\bar{M}^n)$ be closed and $U(\bar{M}^n)$ be open. If $M' \geq M$ then we say that $\bar{a} \in S(M^n) \cap U(M^n)$ is **generic** in $S \cap U$ over $A \subseteq M$ iff we have $d(\bar{a}/A) = \dim(S(M^n) \cap U(M^n))$.

Note 5.1.4. • If $S(\bar{M}^n) \subseteq (\bar{M}^n \setminus M^n)$ then there are no generic elements in $S(\bar{M}^n)$ since generics cannot contain the point ∞ .

- If $\bar{a} \in S(\bar{M}^n)$ is generic over \bar{M} then we do not necessarily have that $d(\bar{a}/\bar{M}) = \dim(S(\bar{M}^n))$ since we may have that $\dim(S(\bar{M}^n)) > \dim(S(\bar{M}^n \cap M^n))$. This does not happen when $S(\bar{M}^n)$ is analytic.
- We also note that, as the types of closed relations giving irreducible sets clearly cannot imply non-trivial disjunctions, the notion of relations being forced makes sense for irreducible sets.

Lemma 5.1.5. If C is a closed relation and $\bar{a} \in C(\bar{M}^n)$ is generic over \bar{M} then $\bar{M}\bar{a} \leq \bar{M}$.

Proof. Let $\dim(C(M^n)) = d(\bar{a}/M) = m$. Note that, by genericity, we actually have that $\delta(\bar{a}/M) = m$, since otherwise $\delta(\bar{a}/M) \geq m$, and since C is quantifier-free we could then have $\bar{a}' \in C(M^n)$ with $\delta(\bar{a}/M) = \delta(\bar{a}'/M) = d(\bar{a}'/M) > m$, contradicting $\dim(C(M^n)) = m$. Thus $d(\bar{a}/M) = \delta(\bar{a}/M)$ i.e. $\bar{M}\bar{a} \leq M'$. \square

Lemma 5.1.6. *If $\bar{a} \in S(\tilde{M}^n)$ is generic over M , where S is special closed, then the only basic closed relations that \bar{a} satisfies with elements of M are those in the definition of $S(\bar{x})$.*

Proof. First assume that $S(M^n)$ is defined over $\bar{c} \subseteq_{fin} M$. As $S(-)$ is special closed, any equality between the coordinates of \bar{a} and M not forced by the definition of $S(\bar{x})$ would reduce the dimension, and so, by the genericity of \bar{a} there cannot be any such equalities. Similarly, there can be no extra H-relations amongst the co-ordinates of \bar{a} since these would make $r(\bar{a}) > |htp(\bar{x}/S)|$ and thus $d(\bar{a}/M) < \max\{d(\bar{x}/M) \mid M' \models S(\bar{x})\}$, contradicting \bar{a} 's genericity.

So the only extra basic closed relations that \bar{a} could satisfy are H-relations with say $c_1, c_2 \in M \setminus \{\bar{a}\}$. We note that we can find $\bar{a}' \in S(M^n)$ generic in $S(\bar{x})$ over M such that \bar{a}' satisfies no such extra H-relations with elements of M . This means that $\dim(S(M^n)) = d(\bar{a}'/M) = \delta(\bar{a}'/\bar{c})$ and also that $\delta(\bar{a}) = \delta(\bar{a}')$ and $r(\bar{a} \leftrightarrow \bar{c} c_1 c_2) > r(\bar{a}' \leftrightarrow \bar{c} c_1 c_2) = r(\bar{a}' \leftrightarrow \bar{c})$. We now see that:

$$\begin{aligned} d(\bar{a}/M) &\leq \delta(\bar{a}/M) \leq \delta(\bar{a}/\bar{c} c_1 c_2) \\ &= \delta(\bar{a}) - r(\bar{a} \leftrightarrow \bar{c} c_1 c_2) \\ &< \delta(\bar{a}') - r(\bar{a}' \leftrightarrow \bar{c}) \quad (\text{by definition of } \bar{a}') \\ &= \dim(S(M^n), \bar{c}) \end{aligned}$$

so that \bar{a} is not generic in $S(M')$ over M . This contradicts our definition of \bar{a} , and so there are no such relations. \square

Proposition 5.1.7. *Let $S(M^n)$ be special closed of dimension greater than 0 and \bar{a} from an elementary extension, M' of M be generic in $S(\bar{x})$ over M . Then, for any $\bar{a}_0 \in S(M^n)$, there is a specialisation $\pi : \tilde{M} \rightarrow \bar{M}$ such that $\pi(\bar{a}) = \bar{a}_0$ and $\pi|_{\bar{M}} = id_{\bar{M}}$ (where $\tilde{M} = M' \cup \{\infty\}$ and $M' \succeq M$).*

Proof. We have, by the lemma above that for any basic closed T such that $M' \models T(\bar{a})$ we have $S(M) \subseteq T(M)$ and so $M \models T(\bar{a}_0)$ for any $\bar{a}_0 \in S(M^n, \bar{c})$. Thus there is a partial specialisation $\pi : M\bar{a} \rightarrow M$ such that $\pi(\bar{a}) = \bar{a}_0$ for any $\bar{a}_0 \in S(M)$.

By the Lemma 5.1.5 $M\bar{a} \leq M'$ and so, by Lemma 4.3.2 we can construct a specialisation $\pi : \tilde{M} \rightarrow \bar{M}$ such that $\pi(\bar{a}) = \bar{a}_0$ and $\pi|_{\bar{M}} = id_{\bar{M}}$. \square

Lemma 5.1.8. *If S is strongly irreducible in some U open in \bar{M}^n such that $S(\bar{M}^n) \cap U(\bar{M}^n) \not\subseteq \bar{M}^n \setminus M^n$, and \bar{a} is generic in $S(\bar{M}) \cap U(\bar{M})$ over \bar{M} then $d(\bar{a}/M) = \dim(S(\bar{M}) \cap U(\bar{M}))$.*

Proof. We have $d(\bar{a}/M) = \dim(S(\bar{M}) \cap U(\bar{M}) \cap M^n)$ and want $d(\bar{a}/M) = \dim(S(\bar{M}) \cap U(\bar{M}))$. This is clear since if $\dim(S(\bar{M}) \cap U(\bar{M}))$ were not realised by some $\bar{a} \in S(\bar{M}) \cap U(\bar{M}) \cap M^n$ then it would have to be realised by some $\bar{b} \in S(\bar{M}) \cap U(\bar{M}) \cap (\bar{M}^n \setminus M^n)$. But then $S(\bar{M}) \cap U(\bar{M}) \cap (\bar{M}^n \setminus M^n)$ is a proper closed subset of $S(\bar{M}) \cap U(\bar{M})$ with the same dimension, and so $S(\bar{M}) \cap U(\bar{M})$ is not strongly irreducible, a contradiction. \square

Proposition 5.1.9. *For S closed and U open in \bar{M}^n such that $S(\bar{M}^n) \cap U(\bar{M}^n) \not\subseteq \bar{M}^n \setminus M^n$ the following are equivalent:*

1. $S \cap U$ is strongly irreducible in U

2. S is definable and for any $\tilde{M} \succeq \bar{M}$, any $\bar{a} \in S(\tilde{M}^n) \cap U(\tilde{M}^n)$ generic over any $A \subseteq \tilde{M}$, and any $\bar{a}_0 \in S(\bar{M}) \cap U(\bar{M})$ there is a specialisation $\pi : \tilde{M} \rightarrow A$ such that $\pi(\bar{a}) = \bar{a}_0$ and π is the identity on A .

Proof. We prove (1) \Rightarrow (2) first by the contrapositive, so assume that there is $\bar{a} \in S(\tilde{M}) \cap U(\tilde{M})$, generic in $S \cap U$ over $A \supseteq \bar{M}$, and $\bar{a}_0 \in S(\bar{M}) \cap U(\bar{M})$ such that there is no specialisation $\pi : \tilde{M} \rightarrow A$ with $\pi(\bar{a}) = \bar{a}_0$. Then there must be a closed $T(\bar{M}')$ such that $\bar{a} \in T(\bar{M}')$ but $\bar{a}_0 \notin T(\bar{M}')$, since otherwise we could put $\pi(\bar{a}) = \bar{a}_0$, $\pi(x) = x$ for all $x \in \bar{M}$ and preserve closed relations, and then extend to a specialisation by induction as in Lemma 4.3.2. Now $T(\tilde{M}) \cap S(\tilde{M}) \cap U(\tilde{M})$ is closed in $U(\tilde{M})$, a proper subset of $S(\tilde{M}) \cap U(\tilde{M})$, and contains \bar{a} , so by the lemma above must have dimension equal to that of $S(\tilde{M}) \cap U(\tilde{M})$. But this means that $S(\tilde{M})$ is not strongly irreducible in U .

For the other direction we assume that for any $\bar{a} \in S(\tilde{M}^n) \cap U(\tilde{M}^n)$ generic over A , and any $\bar{a}_0 \in S(\bar{M}) \cap U(\bar{M})$ there is a specialisation $\pi : \tilde{M} \rightarrow A$ such that $\pi(\bar{a}) = \bar{a}_0$. There clearly cannot be any closed set $T(\bar{M}^n)$ such that $T(\bar{M}^n) \cap U(\bar{M}^n) \subsetneq S(\bar{M}) \cap U(\bar{M})$ with equal dimension, since we could then take $\bar{a} \in T(\tilde{M}^n) \cap U(\tilde{M}^n)$ generic in $S \cap U$ over A and $\bar{a}_0 \in (S(\bar{M}) \cap U(\bar{M})) \setminus (T(\bar{M}^n) \cap U(\bar{M}^n))$, which would contradict π above being a specialisation. \square

Corollary 5.1.10. *All special closed sets of dimension greater than zero are strongly irreducible in any $U \subseteq_{op} M^n$.*

Proof. Immediate from last two results. \square

Corollary 5.1.11 (B1) and (B7). *Open sets are strongly irreducible (as they are special closed in themselves) and thus they are analytic in themselves.*

Proposition 5.1.12. *Let S be closed and O be open set in M^n . Then the following are equivalent:*

1. S is strongly irreducible in O ;
2. S is definable and irreducible in O ;
3. S is special closed of dimension > 0 in O or is a singleton.

Remark 5.1.13. This only holds in open subsets of M^n , and does not hold in all open subsets of \bar{M} . In \bar{M} itself, for instance, we can easily find closed sets which are not irreducible, but cannot be expressed as the union of proper closed subsets. For example, the special closed set $S(\bar{M}^3) = \{\langle x, y, z \rangle \in \bar{M}^3 \mid R(x, y, z)\}$ is reducible since $\dim(S(\bar{M}^3)) = 2 = \dim(\bar{M}^3 \setminus M^3)$, which is a proper closed subset, but there is no decomposition of $S(\bar{M}^3)$ into 2 closed subsets. This is not a problem as long as it does not happen for sets which are analytic in \bar{M} .

Proof. The implication (1) \Rightarrow (2) is clear— $S \cap O$ is irreducible iff it has no proper subsets which are closed in O and of equal dimension, and then it clearly cannot be expressed as a union of 2 proper subsets closed in O , since any such union will have strictly lower dimension than $S \cap O$.

To prove (2) \Rightarrow (3) assume that (2) holds.

First observe that the only zero dimensional sets $S \cap O$ with the property (2) are the singletons, since all quantifier-free definable zero dimensional sets are closed. So we easily get the result for zero dimensional sets, since singletons are in the form required by (3) with $C(M^n) = \{\bar{c}\}$.

We show that if, for any projection $pr : M^n \rightarrow M^l$ we have $\dim(pr(S \cap O)) = 0$ and $|pr(S \cap O)| \geq 2$, then there are sets S_1 and S_2 closed in O such that $S_i \subsetneq S \cap O$ and $S \cap O = S_1 \cup S_2$, contradicting (2). Let $pr : M^n \rightarrow M^l$ be such a projection and let $pr(S \cap O) = U$. Let $|U| \geq 2$. Then U is zero dimensional, and by the maximality of l it is also quantifier free definable (see the proof of [Proposition 4.1.15](#)). Thus, given any $\bar{a} \in U$ we get that both $U \setminus \bar{a}$ and \bar{a} are closed. Thus we get that

$$\begin{aligned} pr^{-1}(U \setminus \bar{a}) \cap S \cap O &= ((U \setminus \bar{a}) \times M^{n-l}) \cap S \cap O \\ \text{and } pr^{-1}(\bar{a}) \cap S \cap O &= (\bar{a} \times M^{n-l}) \cap S \cap O \end{aligned}$$

are both closed. Since $|U| \geq 2$ we also have that $pr^{-1}(U \setminus \bar{a}) \cap S \cap O \subsetneq S \cap O$ and $pr^{-1}(\bar{a}) \cap S \cap O \subsetneq S \cap O$ (i.e. they are both proper subsets). Finally, it is clear that $S \cap O = (pr^{-1}(U \setminus \bar{a}) \cap S \cap O) \cup (pr^{-1}(\bar{a}) \cap S \cap O)$. So, we can assume that $pr(S \cap O)$ is a singleton for all projections pr such that $\dim(pr(S \cap O)) = 0$.

Now, by [Corollary 4.2.17](#), since $O \subseteq M^n$ we can write $S \cap O$ as a finite union of sets of the form $(\hat{T}(M^n) \cap (U(M^{n-l}) \times M^l)) \cap O$ up to permutation of variables, where $\hat{T}(M^n)$ is special closed of dimension ≥ 1 , $U(M^{n-l})$ is zero dimensional special closed, and $1 \leq l \leq n$. Since we are assuming that (2) holds we can clearly assume that it is a trivial union, so that $S \cap O = (\hat{T}(M^n) \cap (U(M^{n-l}) \times M^l)) \cap O$ for $\hat{T}(M^n)$ and $U(M^{n-l})$ as above. We have proved above that all zero dimensional projections are singletons, and so we must have $U(M^{n-l}) = \{\bar{b}\}$ for some point $\bar{b} \in M^{n-l}$. So $S \cap O = \hat{T}(M^n) \cap (\{\bar{b}\} \times M^l) \cap O$ and all zero dimensional projections are singletons. But if we now take the simple closed set given by $C := T \cap \bar{x} = \bar{b}$ we see that $\hat{C}(M^n) = \hat{T}(M^n) \cap (\{\bar{b}\} \times M^l)$, and we have the projection condition, so we have (3).

To complete the proof we need (3) \Rightarrow (1). We get this by assuming (3), and then applying [Lemma 5.1.7](#) and then [Lemma 5.1.9](#). \square

Proposition 5.1.14. $S_1 \subseteq_{\text{irred}} U$ and $S_2 \subseteq_{\text{irred}} V$ iff $S_1 \times S_2 \subseteq_{\text{irred}} U \times V$.

Proof. The left to right direction is clear by considering specialisations and using [Lemma 5.1.9](#). Take $\langle a, b \rangle \in \tilde{M}^{n+m}$ generically satisfying $S_1 \times S_2$ over \bar{M} , and $\langle a_0, b_0 \rangle \in S_1 \times S_2(\bar{M}^{n+m})$. Then we claim that a generically satisfies S_1 over $\bar{M}b$ and that b generically satisfies S_2 over $\bar{M}a$. If not then $d(a/\bar{M}b) < \dim(S_1)$ or $d(b/\bar{M}a) < \dim(S_2)$ and this gives $d(ab/\bar{M}) = \delta(ab/\bar{M}) = \delta(a/\bar{M}b) + \delta(b/\bar{M}) < \dim(S_1) + \dim(S_2) = \dim(S_1 \times S_2)$ or $d(ab/\bar{M}) = \delta(ab/\bar{M}) = \delta(a/\bar{M}b) + \delta(b/\bar{M}) < \dim(S_1) + \dim(S_2) = \delta(b/\bar{M}a) = \dim(S_1 \times S_2)$ contradicting the genericity of $\langle a, b \rangle$. Now by [Lemma 5.1.9](#) there is a specialisation $\pi_1: \bar{M}ab \rightarrow \bar{M}a$ such that $\pi_1(b) = b_0$ and another specialisation $\pi_2: \bar{M}a \rightarrow \bar{M}$ such that $\pi_2(a) = a_0$ (since a 's genericity over $\bar{M}b$ certainly implies its genericity over \bar{M}). We can combine these to get a specialisation $\pi_2 \circ \pi_1: \bar{M}ab \rightarrow \bar{M}$ such that $\pi(\langle a, b \rangle) = \langle a_0, b_0 \rangle$. By [Lemma 5.1.9](#) again we have that $S_1 \times S_2$ is irreducible in $U \times V$.

To get the other direction suppose that one of S_1 and S_2 is not irreducible (in U or V respectively). Without loss of generality suppose that $S'_1 \subsetneq S_1$ is such that

$\dim(S'_1) = \dim(S_1)$. Then $S'_1 \times S_2 \subsetneq S_1 \times S_2$ and $\dim(S'_1 \times S_2) = \dim(S'_1) + \dim(S_2) = \dim(S_1 \times S_2)$, so that $S_1 \times S_2$ is not irreducible in $U \times V$. \square

5.2. Chows' theorem

Theorem 5.2.1. *Suppose S is closed in \bar{M}^n . Then $S(\bar{M}^n)$ is analytic in \bar{M}^n iff it is defined by equalities alone.*

Remark. This is an analogue of Chows theorem, stating that all sets which are analytic in the whole of projective space are in fact algebraic. The proof comes from the following results.

Proposition 5.2.2 (B2). *If $C_1, C_2 \subseteq_{an} U \subseteq_{op} \bar{M}^n$ and $C_3 \subseteq_{an} V \subseteq_{op} \bar{M}^m$ then $C_1 \cup C_2 \subseteq_{an} U$ and $C_1 \times C_3 \subseteq_{an} U \times V$.*

Proof. For $C_1 \cup C_2 \subseteq_{an} U$ take any $a \in U$. By analyticity there are V_1 and V_2 open in U and $C_1^1, C_1^2, \dots, C_1^{k_1}$ and $C_2^1, C_2^2, \dots, C_2^{k_2}$ irreducible in V_1 and V_2 respectively, such that $C_i \cap V_i = C_i^1 \cup C_i^2 \cup \dots \cup C_i^{k_i}$, for $i = 1, 2$. Note that each $C_1^j \cap V_2$ is irreducible in $V_1 \cap V_2 \subseteq_{op} V_1$, and each $C_2^j \cap V_1$ is irreducible in $V_1 \cap V_2 \subseteq_{op} V_2$, and then consider $V_1 \cap V_2 \subseteq_{op} U$. We get:

$$\begin{aligned} (C_1 \cup C_2) \cap (V_1 \cap V_2) &= ((C_1 \cap V_1) \cap V_2) \cup ((C_2 \cap V_2) \cap V_1) \\ &= \bigcup_{i=1}^{k_1} (C_1^i \cap V_2) \cup \bigcup_{i=1}^{k_2} (C_2^i \cap V_1) \end{aligned}$$

and we are done.

For $C_1 \times C_3 \subseteq_{an} U \times V$ take any $\langle a, b \rangle \in U \times V$. There are $V_1 \subseteq_{op} U$ and $V_3 \subseteq_{op} V$, and $C_1^1, C_1^2, \dots, C_1^{k_1}$ and $C_3^1, C_3^2, \dots, C_3^{k_3}$ irreducible in V_1 and V_3 respectively, such that $C_i \cap V_i = C_i^1 \cup C_i^2 \cup \dots \cup C_i^{k_i}$, for $i = 1, 3$. Now considering $V_1 \times V_3 \subseteq_{op} U \times V$ we get:

$$\begin{aligned} (C_1 \times C_3) \cap (V_1 \times V_3) &= (C_1 \cap V_1) \times (C_3 \cap V_3) \\ &= (C_1^1 \cup C_1^2 \cup \dots \cup C_1^{k_1}) \times (C_3^1 \cup C_3^2 \cup \dots \cup C_3^{k_3}) \\ &= \bigcup_{j=1}^{k_1} \left(\bigcup_{i=1}^{k_3} (C_1^j \times C_3^i) \right). \end{aligned}$$

Now, by Proposition 5.1.14 each $C_1^j \times C_3^i$ is irreducible in $V_1 \times V_3$ and we are done. \square

Proposition 5.2.3. *A type-definable subset $C(M^n) \subseteq M^n$ is analytic in M^n iff $C(M^n)$ is locally \mathcal{L}^* -closed.*

Proof. The left to right direction is clear from the definition of analytic.

For the right to left direction first note that if $C(M^n)$ is closed, and $a \notin C(M^n)$ then, since $M^n \setminus C(M^n)$ is open, and contains a , we can take $M^n \setminus C(M^n) = V_a$ to be a neighbourhood of a , and get $V_a \cap C(M^n) = \emptyset$, an (empty) union of irreducibles. Thus we only need to show that $C(M^n)$ is analytic at each $a \in C(M^n)$.

By [Corollary 4.2.17](#) and we can locally write $C(M^n)$ as a finite union of sets of the form $(U(M^{n-l}) \times M^l) \cap S(M^n)$ where $S(M^n)$ is special closed of dimension ≥ 1 , $U(M^{n-l})$ is zero dimensional special closed and $0 \leq l \leq n$. We have already shown ([Proposition 5.2.2](#)) that unions of analytic sets are analytic, so we only need to show that each intersection $(U \times M^l) \cap S$ is analytic, and then we are done.

So we take any such $a \in (U \times M^l) \cap S$ and claim that there is an open $V_a \ni a$ such that $V_a \cap (U \times M^l) \cap S$ is irreducible in V_a . This clearly gives that $(U \times M^l) \cap S$ is analytic in M^n .

We proceed by induction on l , where $n - l$ is the maximal number of co-ordinates that a zero dimensional projection is onto (i.e. the dimension of the ambient space of U).

Basis. If $l = 0$ then $n - l = n$ so that $(U(M^{n-l}) \times M^l) \cap S(M^n) = U(M^n) \cap S(M^n) = O$ is zero dimensional quantifier free definable. Thus $O \setminus \{a\}$ is also zero dimensional quantifier free, and so is special closed in M^n . Thus $M^n \setminus (O \setminus \{a\}) = (M^n \setminus O) \cup \{a\} = V_a$ is open, and a neighbourhood of $\{a\}$. Also $V_a \cap O = \{a\}$, which clearly, as a singleton, is an irreducible set.

Induction. Suppose that if $l' < l$ then the claim is true for any set $(U' \times M^{l'}) \cap T \ni a$, where U' is zero dimensional special closed and T is special closed of dimension > 0 . Now let $a_0 \subseteq a$ be in U and notice that $U \setminus \{a_0\}$ is zero dimensional quantifier free definable, and so closed. Thus $M^{n-l} \setminus (U \setminus \{a_0\}) = (M^{n-l} \setminus U) \cup \{a_0\}$ is open, as is $V_{a_0} = ((M^{n-l} \setminus U) \cup \{a_0\}) \times M^l$.

Note $V_{a_0} \cap (U \times M^l) \cap S = (\{a_0\} \times M^l) \cap S$.

If for any projection pr such that $\dim(pr(\{a_0\} \times M^l) \cap S(M^n)) = 0$ we have that the projection is a singleton, then the special closed set given by $(\{a_0\} \times M^l) \cap S(M^n)$ is irreducible by [Proposition 5.1.12](#).

Otherwise there is such a projection which is not a singleton, and thus cannot be onto exclusively the first $n - l$ co-ordinates (or it would be a sub-tuple of a_0). Say pr_1 is the maximal such projection, so that $pr_1(\{a_0\} \times M^l) \cap S = U' \subseteq M^{n-l'}$ is zero dimensional and $n - l' \geq n - l$. Assume without loss that pr is onto the first $n - l'$ co-ordinates, and note that by the maximality of $n - l'$ this U' is quantifier free definable. Any existential quantifiers in the definition of U' can only refer to variables of S since this is quantifier free definable and $U' = pr_1(\{a_0\} \times M^l) \cap S$. But if these quantifiers are not automatically satisfied by closure under strong extensions ([Axiom 2](#)) then the tuples which satisfy them must be forced to lower the δ -value of a tuple in U (i.e. not be strong extensions of such a tuple in U). But then we could expand the zero dimensional set U' to U'' , a zero dimensional set covering more co-ordinates of $(\{a_0\} \times M^l) \cap S$. This contradicts the maximality of $n - l'$. Thus $(\{a_0\} \times M^l) \cap S = (U' \times M^{l'}) \cap S$ is as in the induction hypothesis, and by induction there is an open V'_a such that $V'_a \cap (U' \times M^{l'}) \cap S$ is irreducible in V'_a (and thus in any open subset of it).

Now we take $V_a = V'_a \cap V_{a_0}$ and get:

$$\begin{aligned} V_a \cap (U \times M^l) \cap S &= V'_a \cap (V_{a_0} \cap (U \times M^l) \cap S) \\ &= V'_a \cap (\{a_0\} \times M^l) \cap S \\ &= V'_a \cap (U' \times M^{l'}) \cap S \end{aligned}$$

and this is irreducible in V_a . This proves the claim and the Proposition. \square

Corollary 5.2.4. *Given $U \subseteq_{op} M^n$ and definable $S \subseteq \bar{M}^n$, $S \cap U$ is analytic in U iff S is closed.*

Proof. By Proposition 2.3.6 and the proposition above, if S is closed then its restriction to any open subset O of M^n is analytic in O . \square

Lemma 5.2.5. *Given a strongly irreducible $C(\bar{M}^n) \subseteq_{an} U(\bar{M}^n) \subseteq_{op} \bar{M}^n$ the following holds:*

$$\left\{ \begin{array}{l} \text{For any variable } x_i \text{ not forced into an equational ideal by } C \text{ either} \\ \text{there is no special closed relation forced on } x_i \text{ by } C \text{ or } U \vdash x_i \neq \infty. \end{array} \right. \quad (\#)$$

Proof. Note first that $U_l = \{\bar{x} \in \bar{M}^n \mid \bigwedge_{i=l}^n x_i \neq \infty\} \cap U(\bar{M}^n)$ is open for each l .

Then observe that we can assume that C is free since otherwise we could work in a lower dimensional space and replace the bound variables with parameters. Thus, since $C(\bar{M}^n)$ is strong irreducible, we have $\dim(C(\bar{M}^n) \setminus U_l) = \dim(C(\bar{M}^n) \cap \{\bar{x} \mid \bigwedge_{i=l}^n x_i = \infty\}) < \dim(C(\bar{M}^n))$ for any $l \leq n$. Thus we get that

$$\dim(C(\bar{M}^n)) = \dim(C(\bar{M}^n) \cap U_l)$$

for all $l \leq n$, so in particular the dimension of $C(\bar{M}^n)$ is realised in M^n . Since U_l is open, we have by Lemma 2.3.5 (which holds here as dimension extends to closed subsets) that $C(\bar{M}^n) \cap U_l$ is strong irreducible in U_l . Thus, taking $l = 2$ we have

$$\dim(C(\bar{M}^n) \cap U_2) > \dim(C(\bar{M}^n) \cap U_2 \cap x_1 = \infty). \quad (*)$$

Since we are assuming that C is free and analytic we can take a point where the dimension of C is realised and know that in some neighbourhood of this point we have, say, $C(\bar{x}) = S_1(\bar{x}^1) \wedge \cdots \wedge S_k(\bar{x}^k)$ where each S_i is a special closed symbol and each $\bar{x}^i \subseteq \bar{x}$. We assume that condition (#) does not hold so without loss can assume that x_1 is in $\bar{x}^1, \dots, \bar{x}^l$ where $1 \leq l \leq n$, and that $U \not\vdash x_1 \neq \infty$. If we also assume, as we can do, that there are not any H-relations repeated in distinct S_i then we get that

$$\dim(C(\bar{M}^n)) = \dim(C(\bar{M}^n) \cap M^n) = |\bar{x}| - \sum_{i=1}^k \text{htp}(\bar{x}_i / S_i)$$

since the dimension of $C(M^n)$ is the dimension of the ambient space minus the number of H-relations that C forces its variable into.

Similarly, using this notation we get that,

$$\dim\left(\left(C \wedge \bigwedge_{i=2}^n x_i \neq \infty \wedge x_1 = \infty\right)(\bar{M}^n)\right) = |\bar{x}| - 1 - \sum_{i=l+1}^k \text{htp}(\bar{x}_i / S_i)$$

since all of the special closed relations S_1, \dots, S_l are realised by any tuple with $x_1 = \infty$. But, since $l \geq 1$ and $\text{htp}(\bar{x}_i / S_i) \geq 1$ for all i , we have $\sum_{i=1}^k \text{htp}(\bar{x}_i / S_i) \geq 1 + \sum_{i=l+1}^k \text{htp}(\bar{x}_i / S_i)$. Finally this gives $\dim((C \wedge \bigwedge_{i=2}^n x_i \neq \infty)(\bar{M}^n)) = \dim(C) \leq \dim((C \wedge \bigwedge_{i=2}^n x_i \neq \infty \wedge x_1 = \infty)(\bar{M}^n))$, which contradicts (*) above. Thus the condition (#) must hold. \square

Lemma 5.2.6. *Given $C \subseteq_{cl} U \subseteq_{op} \bar{M}^n$ we have $C \subseteq_{an} U$ iff the following condition holds:*

C is locally definable and for any variable x_i such that there is $\bar{c} \in C$ with $c_i = \infty$ there is an open neighbourhood $V_{\bar{c}}$ of \bar{c} such that either there are no special closed relations on this x_i implied by the definition of $C \cap V_{\bar{c}}$, or $C \cap V_{\bar{c}} \vdash_{T_\infty} x_i = \infty$.

Proof. For the left to right direction we assume that $C \subseteq_{an} U$.

If $\bar{a}\bar{c} \in C$ with $\infty \in \bar{c}$ then the condition holds vacuously.

Otherwise suppose that $\bar{c} \in C$ has, without loss of generality, $c_1 = \infty$. Also assume, aiming for a contradiction, that for every $U_{\bar{c}} \subseteq_{op} U$ containing \bar{c} there is some special closed relation on the variable x_1 implied by $C \cap U_{\bar{c}}$, and that $(C \cap U_{\bar{c}})(\bar{x}) \not\vdash_{T_\infty} x_1 = \infty$.

By analyticity there is a $V_{\bar{c}} \subseteq_{op} U$ containing \bar{c} and sets C_1, \dots, C_k closed and strongly irreducible in $V_{\bar{c}}$ such that

$$V_{\bar{c}} \cap C = C_1 \cup \dots \cup C_k.$$

Take $U_{\bar{c}} = V_{\bar{c}}$ so that by assumption we have a special closed relation $S(x_1, \bar{x}')$, with $\bar{x}' \subseteq \bar{x}$ such that $(C \cap V_{\bar{c}})(\bar{x}) \vdash_{T_\infty} S(x_1, \bar{x}')$, and so $C_i \vdash_{T_\infty} S(x_1, \bar{x}')$ for $1 \leq i \leq k$.

Note that we can assume $\bar{c} \in C_1 \cap \dots \cap C_k$, since if $\bar{c} \notin C_k$, say, then we would consider $V'_{\bar{c}} = V_{\bar{c}} \setminus C_k \subseteq_{op} U$ in place of $V_{\bar{c}}$ and get $V'_{\bar{c}} \cap C = C_1 \cup \dots \cup C_{k-1}$.

We claim that $C_i(\bar{x}) \vdash_{T_\infty} x_1 = \infty$ for $1 \leq i \leq k$. Otherwise we have that $C_i \subseteq_{irred} U_{\bar{c}}$, that C_i does not force x_1 into any equational ideals, that $U_{\bar{c}} \not\vdash x_1 \neq \infty$ and that $C_i \vdash_{T_\infty} S(x_1, \bar{x}')$, precisely contradicting the last Lemma.

Now, since $V_{\bar{c}} \cap C = C_1 \cup \dots \cup C_{k-1}$ we get $(V_{\bar{c}} \cap C)(\bar{x}) \vdash_{T_\infty} x_1 = \infty$, contradicting our assumption that the condition does not hold, and this proves the first part of the Lemma.

For the converse we take arbitrary $\bar{c} \in U$ and show, supposing the condition holds, that C is analytic at \bar{c} . If $\bar{c} \in M^n$ then we are OK by Proposition 5.2.3, and if $\bar{c} \notin C$ then we can take $V_{\bar{c}} = U \setminus C$. Otherwise we have $\infty \in \bar{c} \in C$, and we can suppose without loss that $c_1, c_2, \dots, c_{n-l} = \infty$, and $c_i \neq \infty$ for $i > n-l$. We note that we can assume that $(V_{\bar{c}} \cap C)(\bar{x}) \not\vdash_{T_\infty} x_i = \infty$ for $i = 1, 2, \dots, n-l$ and any $V_{\bar{c}} \subseteq_{op} U$ containing \bar{c} since if it did then we would ignore any special closed relations involving x_i (they would automatically be satisfied) and so ignore x_i altogether, and consider $C' = \{(x_1, \dots, x_{i-1}, x_{i+1} \dots x_n) \mid \langle x_1, \dots, x_{i-1}, \infty, x_{i+1} \dots x_n \rangle \in C\}$ in place of C in this $V_{\bar{c}}$.

So, supposing that the condition holds we get $V_{\bar{c}} \subseteq_{op} U$ containing \bar{c} such that $V_{\bar{c}} \cap C$ proves no special closed relations on x_i with $i = 1, 2, \dots, n-l$. We can also assume that $V_{\bar{c}} \vdash_{T_\infty} \bigwedge_{i=n-l+1}^n (x_i \neq \infty)$ so that $V_{\bar{c}} \subseteq \bar{M}^{n-l} \times M^l$.

Thus we have that $V_{\bar{c}} \cap C = P \times S$ where P is defined by equality alone, and so analytic in \bar{M}^{n-l} , and S is closed, and so analytic in M^l . Hence by Proposition 5.2.2, $C \subseteq_{an} U \subseteq_{op} \bar{M}^{n-l} \times M^l$. \square

Proof of Theorem. Clearly if $S \subseteq_{cl} \bar{M}^n$ is defined by equalities alone then it can be expressed as a finite union of irreducible sets, and so is analytic at all points. Conversely, if the definition of $S \subseteq_{cl} \bar{M}^n$ locally forces a special closed relation $(C(x_1, \dots, x_l))$ say on $\langle s_1, \dots, s_n \rangle \in S$ then $\langle s_1, \dots, s_n \rangle$ is in S by virtue of satisfying $C(x_1, \dots, x_l)$. This implies that there must be some variable from x_1, \dots, x_l which is not locally forced into an equational ideal by S and we assume that it is x_1 . Thus we have, for any open U

containing s that $S \cap U \not\models x_1 = \infty$. Now note that we cannot have $S \vdash x_1 \neq \infty$ since closed sets are positive, and thus that $\bar{s}^* = \langle \infty, s_2, \dots, s_n \rangle$ is in S by virtue of satisfying $C(x_1, \dots, x_l)$. Then for any open $U_{\bar{s}^*} \ni \bar{s}^*$ we have that $S \cap U_{\bar{s}^*} \vdash C(x_1, \dots, x_l)$. By Lemma 5.2.6 S is not analytic in \bar{M}^n . \square

Corollary 5.2.7 (B3). *If $C_1, C_2 \subseteq_{an} U \subseteq_{op} \bar{M}^n$ then $C_1 \cap C_2 \subseteq_{an} U$.*

Proof. Let $c \in C_1 \cap C_2$. We can assume, without loss of generality, that $c_1, \dots, c_l = \infty$ and that $c_{l+1}, \dots, c_n \neq \infty$. Then there is, by the analyticity of C_1 , an open $V_c \ni c$ such that $C_1 \cap V_c(\bar{x}) \not\models_{T_\infty} S(\bar{x}')$ for all special closed relations $S(\bar{x}')$ on sub-tuples \bar{x}' containing any of x_1, \dots, x_l . By the analyticity of C_2 , there is an open $V'_c \ni c$ such that $C_2 \cap V'_c(\bar{x}) \not\models_{T_\infty} S(\bar{x}')$ for all special closed relations $S(\bar{x}')$ as above. Thus $(C_1 \cap C_2) \cap (V_c \cap V'_c) \not\models_{T_\infty} S(\bar{x}')$ for all such special closed relations $S(\bar{x}')$, since $(C_1 \cap C_2) \cap (V_c \cap V'_c) \equiv (C_1 \cap V_c) \cap (C_2 \cap V'_c)$ and so if it did entail any such special closed relation then we'd have that $(C_1 \cap V_c)$ or $(C_2 \cap V'_c)$ entailed the relation, a contradiction. Thus, by Lemma 5.2.6, $C_1 \cap C_2$ is analytic in U . \square

Corollary 5.2.8. *If $S \subseteq_{an} U$ and $S = C_1 \cup C_2$ where $C_i \neq S$ and $C_i \subseteq_{cl} U$ for $i = 1, 2$ then there are S_1, S_2 analytic in U such that $S_i \neq S$ and $S_1 \cup S_2 = S$. Contrapositively, if S is analytic irreducible then it is closed irreducible.*

Proof. First note that, by induction on the dimension of the ambient space we can assume that S , and hence C_1 and C_2 , are free.

Now, by Lemma 4.2.18 we have that $S = (O \times C) \cap U$ where $O \subseteq \bar{M}^l$ is a conjunction of zero dimensional special closed sets and C is an \mathcal{L}^* -closed set, and so quantifier free definable by 4.2.13. Thus we can assume that $C_i = O_i \times C'_i$ for $i = 1, 2$ with O_i and C'_i as O and C . Since S is analytic we have that none of the first l co-ordinates (those of the special closed O) can go to ∞ and so can assume that $U \subseteq M^l \times \bar{M}^{n-l}$. Then note that for any $\bar{a}' \in O$ we have that $O \setminus \{\bar{a}'\}$ is also closed (taking \bar{a}' away from any of the special closed sets of which O is an intersection). Thus $V_{\bar{a}'} = U \setminus ((O \setminus \{\bar{a}'\}) \times \bar{M}^{n-l})$ is open and $V_{\bar{a}'} \cap S = \{\bar{a}'\} \times C$. Thus we can locally isolate the zero dimensional part of any tuple in S and so O is analytic.

We have, by 5.2.4 that any set closed in U is analytic at any point $\bar{a} \in U \cap M^n$. So say that there is $\bar{a} \in U$ with $\bar{a} = \bar{a}' \frown \langle a_1, \dots, a_n \rangle$ and, without loss, $a_1, \dots, a_l = \infty$ but $a_{l+1}, \dots, a_n \neq \infty$. Then we have, since S is analytic and by Lemma 5.2.6, an open $V_{\bar{a}}$ such that $S \cap V_{\bar{a}}$ forces no special closed relations on x_i for $i = 1, \dots, l$. Also by the paragraph above we can assume that $V_{\bar{a}} \cap S = \{\bar{a}'\} \times (C'_1 \cup C'_2)$, so we just need to find analytic replacements for the C'_i .

Since the C'_i are definable we can write them as $C'_1 = \bigcup_{i=1}^{k_1} T_1^i$ and $C'_2 = \bigcup_{i=1}^{k_2} T_2^i$ where the T_j^i are (non-trivial) conjunctions of equational ideals and special closed relations. Then we have that

$$S \cap V_{\bar{a}} = (C_1 \cap V_{\bar{a}}) \cup (C_2 \cap V_{\bar{a}}) = \bigcup_{i=1}^{k_1} ((\{\bar{a}'\} \times T_1^i) \cap V_{\bar{a}}) \cup \bigcup_{i=1}^{k_2} ((\{\bar{a}'\} \times T_2^i) \cap V_{\bar{a}}),$$

and by the last paragraph that this does not force any special closed relations on x_i for $i = 1, \dots, l$.

Thus, for any $(\{\bar{a}'\} \times T_1^i) \cap V_{\bar{a}}$ which does force such a relation there must be T_2^j such that $(\{\bar{a}'\} \times T_1^i) \cap V_{\bar{a}} \subseteq (\{\bar{a}'\} \times T_2^j) \cap V_{\bar{a}}$ and $(\{\bar{a}'\} \times T_2^j) \cap V_{\bar{a}}$ does not force any such relation (otherwise the disjunction would). But since the T_1^i and T_1^j are just conjunctions of positive relations this gives $T_1^i \subseteq T_2^j$ and that T_2^j forces no special closed relations on x_i for $i = 1, \dots, l$.

So suppose that $T_k^1, \dots, T_k^{m_k}$ force such relations, for $k = 1, 2$ and the other T_k^j (for $j > m_k$) do not. Then for each $i \leq m_1$ we can find some $j > m_2$ such that $T_1^i \subseteq T_2^j$, and for each $i \leq m_2$ we can find some $j > m_1$ such that $T_2^i \subseteq T_1^j$. Then, putting $S'_1 = \bigcup_{i=m_1+1}^{k_1} T_1^i$ and $S'_2 = \bigcup_{i=m_2+1}^{k_2} T_2^i$ we get $C'_1 \cup C'_2 = S'_1 \cup S'_2$. Then, by Lemma 5.2.6 the S'_i are analytic at \bar{a} and any point in U with any of the first l co-ordinates $= \infty$.

Repeating this process for any co-ordinate x_i such that there is $\bar{a} \in U$ with distinct $a_i = \infty$ (a finite process) we arrive at S'_i which are analytic at all of their points and such that $S'_1 \cup S'_2 = C'_1 \cup C'_2$. Putting $S_i = O_i \times S'_i$ we see that the S_i are also analytic in U and $S_1 \cup S_2 = S$. \square

Proposition 5.2.9 (C4). *If $S \subseteq_{an} U \subseteq_{op} \bar{M}^n$ is such that S is irreducible, and $V \subseteq_{op} U$ then $S \cap V$ is irreducible in V , and if it is non-empty then $\dim(S \cap V) = \dim(S)$.*

Proof. The proof of the first part goes through exactly as in Proposition 2.3.9 using Corollary 5.2.8 in place of 2.3.8.

Since S is irreducible in U , by Lemma 2.3.11 we can take any $a \in S$ realising the dimension of S and find an open set $U_a \ni a$ such that $S \cap U_a = C$ is strongly irreducible in U_a . Then for any $S_1 \subseteq_{cl} \bar{M}^n$ we have $S_1 \cap C = C$ or $\dim(S_1 \cap C) < \dim(C)$. Thus given any $V \subseteq_{op} \bar{M}^n$ we can let $S_1 = \bar{M}^n \setminus V$ and we get $V \cap C = \emptyset$ (if $S_1 \cap C = C$) or $\dim(V \cap C) = \dim(C)$ (if $\dim((\bar{M}^n \setminus V) \cap C) = \dim(S_1 \cap C) < \dim(C)$). If there is some a realising the dimension of S such that the second possibility holds then $\dim(V \cap S) = \dim(S)$. Otherwise the first possibility holds for every a realising the dimension of S and so a tuple which realises only the H-relations and equalities forced by S (i.e. one realising its dimension) cannot exist in V . This means that one of the (positive) relations forced by S is negated in V and so V contradicts S , so that $V \cap S = \emptyset$. \square

Proposition 5.2.10 (C5). *If $S \subseteq_{an} U \subseteq_{op} P^n$ and S is irreducible in U then for any $S_1 \subsetneq S$, $S_1 \subseteq_{cl} U$, we have $\dim S_1 < \dim S$.*

Proof. We use the contrapositive so suppose that there is a set $S' \subsetneq S$ closed in U such that $\dim(S') = \dim(S)$. We will show that there are closed sets C^* and $C^\# \subsetneq S$ such that $S = C^\# \cup C^*$, and then Lemma 5.2.8 shows that they are analytic.

If there is an open set $V \subseteq_{op} U$ such that $V \cap S = V \cap S' \neq \emptyset$ consider $U \setminus V = S'' \subseteq_{cl} U$. Then $S = S' \cup (S'' \cap S)$ is the decomposition we require.

If there is no such V then for any $V \subseteq_{op} U$ we have $V \cap S' \subsetneq V \cap S$, and that if $V \cap S' \neq \emptyset$ then $\dim(V \cap S') = \dim(V \cap S)$ by Proposition 5.2.9. Take any $a \in S' (\subseteq S)$ and by analyticity find some $V \subseteq_{op} U$ containing a (so $V \cap S \neq \emptyset$) sufficiently small so that $V \cap S = (V \cap C_1) \cup \dots \cup (V \cap C_k)$, where each $V \cap C_i$ is strongly irreducible in V . We note that $k \neq 1$ since otherwise $V \cap S'$ contradicts the strong irreducibility of $V \cap C_1$. Thus, since for each i there is $a \in (C_i \setminus \bigcup_{j \neq i} C_j) \cap V$ and every $S \cap C_i$ is a subset of S

so that in fact $a \in (C_i \cap S) \setminus \bigcup_{i \neq j} (C_j \cap S)$ we get that $C = \bigcup_{i=1}^k (C \cap C_i) \cup (C \cap (U \setminus V))$ is a non-trivial decomposition. \square

Corollary 5.2.11. *If $C \subseteq_{an} U \subseteq_{op} P^n$ then C is strongly irreducible in U iff it is definable and irreducible in U .*

Proof. Immediate from the definitions and the proposition above. \square

Corollary 5.2.12. *Given $C \subseteq_{an} U \subseteq_{op} M^{n-l} \times \bar{M}^l$ (with l maximal) we have that C is irreducible in U iff $C \cong (S(M^{n-l}) \times I(\bar{M}^l)) \cap U$ for some equational ideal I and special closed $S(M^n)$, where \cong means ‘= up to permutation of co-ordinates’.*

Proof. By the Lemma 5.2.6 if $C = (S(M^{n-l}) \times I(\bar{M}^l)) \cap U$ and $C \subseteq_{an} U$ then there is no $\bar{c} \in C$ with $c_i = \infty$ for $i = 1, \dots, n-l$. Then by Proposition 5.1.12 $S(M^{n-l})$ is irreducible in M^{n-l} and clearly $I(\bar{M}^l)$ is irreducible in \bar{M}^l so that $(S(M^{n-l}) \times I(\bar{M}^l))$ is irreducible in $M^{n-l} \times \bar{M}^l$ and any open subset of it by Proposition 5.1.14.

Conversely, if $l = 0$, then by Proposition 5.1.12 C is of the form required. Otherwise $l > 0$ and by its minimality $\exists \bar{c} \in C \cap U$ with $c_i = \infty$. Note that, as above, we can assume that $(C \cap V_{\bar{c}})(\bar{x}) \not\models_{T_\infty} x_i = \infty$ for any open $V_{\bar{c}}$ containing \bar{c} since if it did then we would ignore any special closed relations involving x_i (they would automatically be satisfied) and so ignore x_i altogether, and consider $C' \cap V_{\bar{c}} = \{\langle x_1, \dots, x_{i-1}, x_{i+1} \dots x_n \rangle \mid \langle x_1, \dots, x_{i-1}, \infty, x_{i+1} \dots x_n \rangle \in C \cap V_{\bar{c}}\}$ in place of $C \cap V_{\bar{c}}$. Since we are assuming $C \cap U$ is irreducible in U we can clearly assume that there are no non-trivial disjunctions in the type which defines C , and can speak of special closed relations being forced by C . Thus Lemma 5.2.6 gives us that there is an open $V_{\bar{c}} \ni \bar{c}$ such that $C \cap V_{\bar{c}}$ does not force any special closed relation on x_i . Thus we can only have x_i in equalities, and since we have no disjunctions in C we only have x_i in a single equational ideal. This holds for any variable x_i such that $c_i = \infty$ so, without loss, we let these be the last l variables. Then, by Propositions 5.1.12 and 5.1.14 we get that C is locally of the form required.

It only remains to prove that we cannot have 2 distinct such expressions for $C \cap U_1$ and $C \cap U_2$ in distinct open neighbourhoods U_1 and U_2 . But then we can take one of the two (U_1 say) to contain a generic point and then let $T = U \setminus U_2$, which is closed in U , and get that $C' = C \cap T$ is a proper closed subset of C containing a generic point, and thus has the same dimension. This contradicts the irreducibility of C by Proposition 5.2.10. \square

Corollary 5.2.13. *Irreducible analytic sets are \mathcal{L}^* -closed and thus definable. Thus by Corollary 5.2.11 the irreducible analytic sets and strongly irreducible closed sets coincide.*

5.3. Decomposition into irreducible sets

Proposition 5.3.1 (B6). *If $S \subseteq_{an} U \subseteq_{op} \bar{M}^n$ and $a \in S$ then there is $S_a \subseteq_{an} U$, a finite union of sets irreducible in U containing a , and $S' \subseteq_{an} U$ such that $a \in S_a \setminus S'$ and $S = S_a \cup S'$*

Proof. If S is irreducible then take $S_a = S$ and $S' = \emptyset$.

Otherwise, by the analyticity of S we get $V_a \subseteq_{op} U$ and S_1, \dots, S_k closed in U such that the $S_i \cap V_a$ are all strongly irreducible in V_a and $S \cap V_a = (S_1 \cap V_a) \cup \dots \cup (S_k \cap V_a)$.

Note that we can take k minimal and thus assume that $a \in S_i \cap V_a$ for all i since if $a \notin S_k$ then we could take $V'_a = V_a \setminus S_k$ to be a smaller open set containing a , and get $S \cap V'_a = (S_1 \cap V'_a) \cup \dots \cup (S_{k-1} \cap V'_a)$.

Take $S' = (U \setminus V_a) \cap S$, which is a subset of S closed in U , and note that $a \notin S'$ and $S = (S_1 \cap S) \cup (S_2 \cap S) \dots \cup (S_k \cap S) \cup S'$. Then by Lemma 5.2.8 S' and each $(S_i \cap S)$ is analytic in U .

Next note that for any $S_i^* \subseteq_{cl} U$ such that $S_i^* \cap V_a = S_i \cap V_a$, we can replace S_i by S_i^* in the above decomposition. I claim that we can choose S_i^* so that each $S_i^* \cap S$ is irreducible.

We note that, since $S_i \cap V_a$ is strongly irreducible in V_a and $S_i \cap V_a \subseteq S \cap V_a$ we have $(S_i \cap S) \cap V_a = S_i \cap V_a = (I_i(\bar{M}^{n-l}) \times C_i(M^l)) \cap V_a$ for some equational ideal I and special closed $C(M^l)$, up to permutation of co-ordinates, by Corollary 5.2.12.

Thus we can put $(S_i^* \cap S) = (I_i(\bar{M}^{n-l}) \times C_i(M^l)) \cap U$ and get, again from Corollary 5.2.12, that $S_i^* \cap S$ is strongly irreducible. This proves the Proposition since we have $S_a = (S_1 \cap S) \cup (S_2 \cap S) \dots \cup (S_k \cap S)$ a union of sets strongly irreducible in U containing a and $S' \subseteq_{an} U$ such that $a \in S_a \setminus S'$ and $S = S_a \cup S'$. \square

Corollary 5.3.2. *We have all the [Analytic] axioms in the expanded Hrushovski structure \bar{M} .*

Definition 5.3.3. A set C_a containing a , irreducible in U and such that there is a proper subset of C , C' , analytic in U such that $C = C_a \cup C'$ (as in the above proposition) is called an **irreducible component of S containing a** .

Proposition 5.3.4. *Given $a \in C \subseteq_{an} U \subseteq_{op} \bar{M}^n$ the number of irreducible components of C containing a is finite.*

Proof. If there were infinitely many components all containing a and irreducible in U call them S_a^i with $i \in \mathbb{N}$. Then for any $V_a \subseteq_{op} U$ containing a we have $V_a \cap S_a^i \neq \emptyset$, and so by Proposition 2.3.9 $V_a \cap S_a^i$ is irreducible and has dimension $= \dim(S_a^i)$. But there are infinitely many distinct such $V_a \cap S_a^i$, and this contradicts analyticity. \square

Proposition 5.3.5. *Any $C \cap U \subseteq_{an} U \subseteq_{op} \bar{M}^n$ is the union of a finite or countable number of irreducible components.*

Proof. We have from the above that $C = O \times S$ for some zero dimensional O and \mathcal{L}^* -closed S . By the fact that we have the countable closure property (see Section 3.4) O is countable, so writing $O = \{b_i\}_{i \in \mathbb{N}}$ we have $C = \bigcup_{i \in \mathbb{N}} (\{b_i\} \times S)$ and we can consider each $\{b_i\} \times S$ in turn.

We use induction on the number of operations (\times, \cap, \cup) used to get S from special closed sets and equational ideals to show that S is a finite or countable union of irreducible components, and this will clearly do.

Basis: If S is special closed of dimension > 0 or is defined by an equational ideal then since $S \cap U$ is analytic in U we get by Lemma 5.2.6 that $U \subseteq M^n$. Then by 5.1.10 $S \cap U$ is strongly irreducible in U , and this irreducible subset is unique.

If S is special closed such that $S \cap U \cap M^n$ is zero dimensional then $S \cap U \cap M^n$ is countable (by 3.4.7) and each element is an irreducible component. Also

$$S \cap U \cap (\bar{M}^n \setminus M^n) = U \cap (\bar{M}^n \setminus M^n) = \bigcup_{i=1}^n (U \cap \{\bar{x} \in \bar{M}^n \mid x_i = \infty\})$$

is a decomposition of $S \cap U \cap (\bar{M}^n \setminus M^n)$ into finitely many irreducible components. If S is defined by an equational ideal then it is clearly irreducible already.

Induction: Let $S = A * B$ where $*$ $\in \{\times, \cap, \cup\}$. By the induction hypothesis $A = \bigcup_{i \in \mathbb{N}} S_i$ and $B = \bigcup_{i \in \mathbb{N}} C_i$ for unique S_i and C_i irreducible in U . If $*$ $= \cup$ then clearly S is a unique countable union of irreducibles. Also if $*$ $= \times$ then $S = (\bigcup_{i \in \mathbb{N}} S_i) \times (\bigcup_{i \in \mathbb{N}} C_i) = \bigcup_{i,j \in \mathbb{N}} (S_i \times C_j)$, and by Corollary 5.1.14 each $S_i \times C_j$ is irreducible in U , and this decomposition is unique.

If $*$ $= \cap$ then $S = (\bigcup_{i \in \mathbb{N}} S_i) \cap (\bigcup_{i \in \mathbb{N}} C_i) = \bigcup_{i,j \in \mathbb{N}} (S_i \cap C_j)$. We need to show that if S_i and C_j are irreducible in U then $S_i \cap C_j$ is a unique countable union of irreducibles in U . We can assume that $U \subseteq M^n \times \bar{M}^m$, so that $S_i = Y \times I$ and $C_i = Z \times J$ where Y, Z are irreducible in M^n and I, J are irreducible in \bar{M}^m . This means that I, J are equational ideals, and (by Proposition 5.1.12) Y and Z are either singletons or special closed of dimension > 0 and any zero dimensional projection of Y or Z is a singleton. By Corollary 4.1.18 we have that $Y \cap Z$ is a finite union of sets of the form $(O \times M^l) \cap T(M^n)$, up to permutation of co-ordinates, where $O \subseteq M^{n-l}$ is zero dimensional special closed, and $T(M^n)$ is special closed of dimension > 0 .

Now, since $I \cap J$ is an equational ideal, it is irreducible and we only need to show that each set of the form $(O \times M^l) \cap T(M^n)$ is a countable union of irreducibles. We proceed as in Proposition 5.2.3 by induction on l . Since O is of dimension 0 it is countable—write $O = \bigcup_{i \in \mathbb{N}} \{\bar{a}_i\}$. Clearly

$$(O \times M^l) \cap T(M^n) = \bigcup_{i \in \mathbb{N}} (\{\bar{a}_i\} \times M^l) \cap T(M^n).$$

If $l = 0$ then this is simply a countable union of singletons, which are irreducible, and so we are done.

So suppose that $l > 0$ and assume the result holds for sets $(O' \times M^{l'}) \cap T(M^n)$ with O' zero dimensional special closed and $l' < l$. If $\dim(\text{pr}(\{\bar{a}_i\} \times M^l) \cap T(M^n)) > 0$ for all projections which are not onto singletons then we are done, since then $(\{\bar{a}_i\} \times M^l) \cap T(M^n)$ is special closed, and so irreducible by Proposition 5.1.12. Otherwise we can take the zero dimensional projection pr onto the maximal number of co-ordinates, $n - l'$ say, and note that this is $> n - l$. We can also assume without loss that it is onto the first $n - l'$ co-ordinates and say $O' = \text{pr}(\{\bar{a}_i\} \times M^l) \cap T(M^n)$, so that $(\{\bar{a}_i\} \times M^l) \cap T(M^n) = (O' \times M^{l'}) \cap T(M^n)$. Since $n - l'$ is maximal we get, by the same argument as in Proposition 5.2.3, that O' is quantifier free definable and so special closed. Thus, since $l' < l$ and the set is of the correct form, we get by induction that $(O' \times M^{l'}) \cap T(M^n)$ is a countable union of irreducible sets, and thus so is $(\{\bar{a}_i\} \times M^l) \cap T(M^n)$. Hence $(O \times M^l) \cap T(M^n)$ is a countable union of countable unions of irreducible sets. \square

5.4. Dimension, quantifier elimination and smoothness

Here we finish the proof that the expanded Hrushovski structure is an analytic Zariski structure and get 2 extra results.

Proposition 5.4.1 (C3). *For $S \subseteq_{an} U \subseteq_{op} \bar{M}^n$ we have:*

$$\dim(S) = \max\{\dim(S_a) \mid S_a \text{ is an irreducible component of } S\}.$$

Proof. Say S is defined (by a type of \mathcal{L}^* -closed relations) over A and that $\dim(S) = \max\{d(a/A) \mid a \in S\}$ is realised at a_0 . Now, by Proposition 5.3.1 there is an irreducible component of S , S_{a_0} containing a_0 and so $\dim(S_{a_0}) \geq d(a_0/A) = \dim(S)$. We clearly cannot have $\dim(S_a) > \dim(S)$ for any component S_a , since this would contradict the definition of dimension. \square

Proposition 5.4.2 (C6 for Strongly Irreducible Sets). *For strongly irreducible $S \subseteq_{an} U \subseteq_{op} \bar{M}^n$ such that pr is proper on S*

$$\dim(pr(S)) = \dim(S) - \min\{\dim(pr^{-1}(s) \cap S) \mid s \in pr(S)\}.$$

Proof. Let S be defined over A and $\dim(S) = n$, and suppose that

$$\min\{\dim(pr^{-1}(a) \cap S) \mid a \in pr(S)\} = k.$$

This means that for any $a \in pr(S)$ there is some b such that $\langle a, b \rangle \in S$ and $d(b/Aa) \geq k$ (and for any such b , $d(b/Aa) \leq k$). Then for any $a \in pr(S)$ we must have that $d(a/A) \leq n - k$, since otherwise by Proposition 3.1.4 we would have that $d(ab/A) = d(b/Aa) + d(a/A) > k + (n - k) = n$ contradicting $\dim(S) = n$. Thus

$$\dim(pr(S)) \leq n - k = \dim(S) - \min\{\dim(pr^{-1}(a) \cap S) \mid a \in pr(S)\}.$$

Conversely we show that $\min\{\dim(pr^{-1}(a) \cap S) \mid a \in pr(S)\} \geq \dim(S) - \dim(pr(S))$. For any $\bar{a} \in pr(S)$ there is \bar{y} such that $\langle \bar{a}, \bar{y} \rangle \in S$ and since S is quantifier free definable we can assume, for \bar{y} realising the dimension of $\dim(pr^{-1}(a) \cap S)$ that $d(\bar{y}/A\bar{a}) = \delta(\bar{y}/A\bar{a})$. Thus, for any $\bar{a} \in pr(S)$:

$$\begin{aligned} \dim(pr^{-1}(a) \cap S) &= \max\{\delta(\bar{y}/A\bar{a}) \mid \langle \bar{a}, \bar{y} \rangle \in S\} \\ &= \max\{\delta(A\bar{a}\bar{y}) - \delta(A\bar{a}) \mid \langle \bar{a}, \bar{y} \rangle \in S\} \\ &= \max\{\delta(\bar{y}) - r(\bar{y} \leftrightarrow A\bar{a}) \mid \langle \bar{a}, \bar{y} \rangle \in S\}. \end{aligned}$$

But since S is irreducible in U we have by Corollary 5.2.12 that S is of the form $S \cong (\hat{T}(M^{n-l}) \times I(\bar{M}^l)) \cap U \subseteq M^{n-l} \times \bar{M}^l$ where \hat{T} is special closed of dimension > 0 and I is an equational ideal, and \cong means ‘= up to permutation of coordinates’. By 5.2.5 any of the coordinates which are forced into the special closed relation (v_i , $i = 1, \dots, l$ say) must be such that $S \vdash v_i \neq \infty$. We can also assume that they are not fixed by S since if they were then they would be in the equational ideal. But if these same coordinates are not projected onto (so that $v_i \in \bar{y}$) then $pr^{-1}(a) \cap S$ is not closed in \bar{M}^n for any $a \in pr(S)$, since $pr^{-1}(a) \cap S \vdash v_i \neq \infty$. This contradicts the properness of the projection as sets closed in \bar{M}^n cannot restrict their variables from being ∞ without fixing them to some other parameter.

Thus S does not force any of the \bar{y} -variables into special closed relations so that for \bar{y} realising the dimension of $pr^{-1}(\bar{a}) \cap S$ we have $r(\bar{y} \leftrightarrow A\bar{a}) = 0$ and so the above expression for $\dim(pr^{-1}(\bar{a}) \cap S)$ becomes

$$\begin{aligned}
&= \max\{\delta(\bar{y}) \mid \langle \bar{a}, \bar{y} \rangle \in S\} \\
&= \max\{\delta(\bar{y}) - r(\bar{y} \leftrightarrow A\bar{x}) \mid \langle \bar{x}, \bar{y} \rangle \in S\} \\
&= \max\{|\bar{y}| + |A\bar{x}| - (r(\bar{y}) + r(\bar{y} \leftrightarrow A\bar{x}) + r(A\bar{x})) - (|A\bar{x}| - r(A\bar{x})) \mid S(\bar{x}, \bar{y})\} \\
&= \max\{\delta(A\bar{x}\bar{y}) - \delta(A\bar{x}) \mid \langle \bar{x}, \bar{y} \rangle \in S\} \\
&= \max\{\delta(A\bar{x}\bar{y}) - \delta(A) - (\delta(A\bar{x}) - \delta(A)) \mid \langle \bar{x}, \bar{y} \rangle \in S\} \\
&= \max\{\delta(A\bar{x}\bar{y}/A) - \delta(A\bar{x}/A) \mid \langle \bar{x}, \bar{y} \rangle \in S\} \\
&\geq \max\{\delta(A\bar{x}\bar{y}/A) \mid \langle \bar{x}, \bar{y} \rangle \in S\} - \max\{\delta(A\bar{x}/A) \mid \exists \bar{y} \langle \bar{x}, \bar{y} \rangle \in S\} \\
&= \dim(S) - \dim(pr(S)). \quad \square
\end{aligned}$$

Corollary 5.4.3. *Our definition of analyticity is natural and so this gives us axioms [B4], [B5], [C6] and [C7] automatically.*

Proof. By Notes 2.3.20 and 5.1.2, and from Corollary 5.2.13 this is immediate from the last lemma. \square

Corollary 5.4.4. *\bar{M} is a compact analytic Zariski structure.*

Proof. We have all the [Language], [Analytic] and [Dimension] axioms in the expanded Hrushovski structure \bar{M} by the above and previous results. \square

We now prove a stronger form of quantifier elimination, based on our earlier result, Corollary 3.3.2.

Proposition 5.4.5 (Quantifier Elimination). *Any subset of \bar{M}^n definable in \mathcal{L}^* is a Boolean combination of sets of the form $pr(S)$ where $S \subseteq_{an} U$ for some $U \subseteq_{op} \bar{M}^n$ and pr is a standard projection mapping.*

Proof. There is an obvious map from \mathcal{L}^* -formula to $\mathcal{L}(\bar{M})$ -formula, since the language \mathcal{L}^* is defined directly in terms of the language $\mathcal{L}(\bar{M})$. This means that we can transform an \mathcal{L}^* -formula to an $\mathcal{L}(\bar{M})$ -formula, use the quantifier elimination results we have for \mathcal{L} (see Section 3.3), and then transform it back. So, given any \mathcal{L}^* -formula, $\xi_{\mathcal{L}^*}(\bar{x})$, find an $\mathcal{L}(\bar{M})$ -formula, $\xi(\bar{x})$, such that for any $\bar{a} \in \bar{M}^n$ we have that

$$\bar{M} \models \xi_{\mathcal{L}^*}(\bar{a}) \Leftrightarrow \bar{M} \models \xi(\bar{a}).$$

Note that $\xi_{\mathcal{L}^*}$ is quantifier free iff ξ is.

Now restricting to the ‘affine’ space M^n and considering $\xi_{\mathcal{L}^*}(\bar{M}^n) \cap M^n = \xi(\bar{M}^n) \cap M^n$ we have from Corollary 3.3.2 that this can be defined by a Boolean combination of existential $\mathcal{L}(\bar{M})$ -formulas. We can, in fact, assume that it is a conjunction of existential $\mathcal{L}(\bar{M})$ -formulas and their negations, since if there are any disjunctions we can deal with each disjunct individually. So, we have that there are $\mathcal{L}(\bar{M})$ -formulas $\exists y_i \psi_i(\bar{x}, y_i)$ with ψ_i

quantifier free, such that

$$\begin{aligned} \xi(\bar{x}) \wedge \bigwedge_{i=1}^n (x_i \neq \infty) \equiv_{T_\infty} \bigwedge_{i=1}^k \exists y_i \left(\psi_i(\bar{x}, y_i) \wedge \bigwedge_{i=1}^n x_i \neq \infty \right) \\ \wedge \bigwedge_{i=k+1}^l \neg \exists y_i \left(\psi_i(\bar{x}, y_i) \wedge \bigwedge_{i=1}^n x_i \neq \infty \right). \end{aligned} \quad (*)$$

As each ψ_i is quantifier free, and a quantifier free $\mathcal{L}(\bar{M})$ -formula is equivalent to a quantifier free \mathcal{L}^* -formula, we can write, for each $i = 1 \dots l$,

$$\begin{aligned} \psi_i(\bar{x}, y_i) \equiv_{T_\infty} \bigwedge_{j=1}^{k_1} S_{i,j}(\bar{x}^{i,j}, y_i) \wedge I_i(\bar{x}) \\ \wedge \bigwedge_{j=k_1+1}^{k_2} \neg S_{i,j}(\bar{x}^{i,j}, y_i) \wedge \bigwedge_{j=k_2+1}^{k_3} \neg J_{i,j}(\bar{x}) \wedge \bigwedge_{j=1}^n x_j \neq \infty, \end{aligned}$$

where the $S_{i,j}$ are special closed relations, $\bar{x}^{i,j} \subseteq \bar{x}$, and I and the $J_{i,j}$'s are equational ideals over \bar{M} since all quantifier free \mathcal{L}^* -formulae which restrict their realisations to \bar{M} are of this form.

Now, for each $i = 1 \dots l$, let

$$\begin{aligned} U_i = \left\{ \langle \bar{x}, y_i \rangle \in \bar{M}^{n+1} \mid \left(\bigwedge_{j=k_1+1}^{k_2} \neg S_{i,j}(\bar{x}^{i,j}, y_i) \right) \wedge \left(\bigwedge_{j=k_2+1}^{k_3} \neg J_{i,j}(\bar{x}) \right) \right. \\ \left. \wedge \left(\bigwedge_{j=1}^n x_j \neq \infty \right) \right\}. \end{aligned}$$

Since this is clearly the realisation of the negation of a positive formula, and stops the variables from being ∞ , U_i is an open subset of M^{n+1} . Since the rest of the definition of $\psi_i(\bar{M}^{n+1})$ is positive and also quantifier-free, we get that it is \mathcal{L}^* -closed as a subset of U_i , and so by Proposition 5.2.3 we get $\psi_i(\bar{M}^{n+1}) = A_i$ is analytic in U_i .

Thus, re-writing (*) we get that

$$\xi(\bar{M}^n) \cap M^n = \bigcap_{i=1}^k pr(A_i) \setminus \bigcap_{i=k+1}^l pr(A_i),$$

i.e. $\xi(\bar{M}^n) \cap M^n$ is a Boolean combination of projections of sets analytic in open subsets.

We want to show this for $\xi(\bar{M}^n) \cap (\bar{M}^n \setminus M^n)$ also. Clearly $\xi(\bar{M}^n) \cap (\bar{M}^n \setminus M^n) = \bigcup_{i=1}^n (\xi(\bar{M}^n) \cap \{x_i = \infty\})$, so we consider each $\xi(\bar{M}^n) \cap \{x_i = \infty\}$ in turn. Up to permutation of co-ordinates we can write each of these as $B_i \times \{\infty\}$, with $B_i \subseteq \bar{M}^{n-1}$ since any occurrence of x_i in the definition of $\xi(\bar{M}^n)$ need only be replaced by ∞ (e.g. $\xi(\bar{M}^n) \cap \{x_n = \infty\} = \xi(\bar{M}^{n-1}, \infty)$). By induction on n we have that each B_i is a Boolean combination of projections of sets analytic in open subsets and so the whole of $\xi(\bar{M}^n)$ is a Boolean combination of projections of sets analytic in open subsets, as required. \square

Note 5.4.6. We see from this proof that in fact we only need Boolean combinations of definable analytic sets.

Theorem 5.4.7 (Pre-smoothness). *Given $S_1, S_2 \subseteq_{an} U \subseteq_{op} M^{n-l} \times \bar{M}^l$ both irreducible in U , and any irreducible component S of $S_1 \cap S_2$ we have that*

$$\dim(S) \geq \dim(S_1) + \dim(S_2) - n.$$

Proof. By Corollary 5.2.12 $S_i = \widehat{T}_i(M^{n-l}) \times I_i(\bar{M}^l)$, where \widehat{T}_i is the special closed relation (of dimension > 0) obtained from the simple closed T_i , and I_i is an equational ideal. Thus it is easy to count the number of independent relations (H-relations and equalities) forced by each S_i , as it equals $|htp(\bar{x}/T_i)| + |I_i| = k_i$, say. Then $\dim(S_i) = n - k_i$, since the potential number of elements in a tuple from S_i is n minus the number of independent equalities forced by S_i , and to get the dimension we then minus the number of independent H-relations forced by S_i .

Now $S_1 \cap S_2 = (\widehat{T}_1 \wedge \widehat{T}_2)(M^{n-l}) \times (I_1 \wedge I_2)(\bar{M}^l)$ forces $\leq k_1 + k_2$ independent relations, since some of the relations forced by S_1 and S_2 may coincide. Thus, by the same reasoning as above,

$$\begin{aligned} \dim(S_1 \cap S_2) &\geq n - (k_1 + k_2) \\ &= (n - k_1) + (n - k_2) - n \\ &= \dim(S_1) + \dim(S_2) - n. \end{aligned}$$

We now show that for any irreducible component $S \subseteq S_1 \cap S_2$ we have $\dim(S) = \dim(S_1 \cap S_2)$.

Suppose that the projection $pr : \bar{M}^n \rightarrow \bar{M}^k$ is onto some of the first $n - l$ co-ordinates, and is such that $\dim(pr(S_1 \cap S_2)) = 0$, $|pr(S_1 \cap S_2)| > 1$ and k is maximal. Then for any $a \in pr(S_1 \cap S_2)$ we have $\dim(pr^{-1}(a) \cap (S_1 \cap S_2)) = \dim(S_1 \cap S_2)$. To prove this suppose without loss that $pr(x_1, \dots, x_n) = (x_1, \dots, x_k) = \bar{x}'$, and let $\bar{x}'' = (\bar{x} \setminus \bar{x}')$. Then we have that $htp(\bar{x}'/(T_1 \wedge T_2)) = |\bar{x}'|$ (to get 0-dimensionality) and so

$$|htp(\bar{x}'/(T_1 \wedge T_2))| + |htp(\bar{x}''/(T_1 \wedge T_2))| \leq |htp(\bar{x}/(T_1 \wedge T_2))|,$$

and

$$|htp(\bar{x}/(T_1 \wedge T_2))| \leq |htp(\bar{x}/T_1)| + |htp(\bar{x}/T_2)| = k_1 + k_2 - |I_1| - |I_2|.$$

Together these give

$$|htp(\bar{x}''/(T_1 \wedge T_2)(\bar{a}))| \leq k_1 + k_2 - |I_1| - |I_2| - |\bar{x}'|.$$

Thus, since I_1 and I_2 may restrict variables which are not in \bar{x}'' ,

$$\begin{aligned} \dim(pr^{-1}(a) \cap (S_1 \cap S_2)) &\geq (n - |\bar{x}'|) - |htp(\bar{x}''/(T_1 \wedge T_2)(\bar{a}))| - |I_1| - |I_2| \\ &\geq (n - |\bar{x}'|) - (k_1 + k_2 - |I_1| - |I_2| - |\bar{x}'|) \\ &\quad - |I_1| - |I_2| \\ &= n - k_1 - k_2 = \dim(S_1 \cap S_2). \end{aligned}$$

Since $S_1 \cap S_2 = \bigcup_{a \in pr(S_1 \cap S_2)} pr^{-1}(a) \cap (S_1 \cap S_2)$ the following claim will prove the theorem.

Claim. Each $pr^{-1}(a) \cap (S_1 \cap S_2) = S_a$ is a finite union of irreducibles of equal dimension.

Firstly we take the set

$$S'_a = \widehat{C}_a(M^{n-l}) \times (I_1 \wedge I_2)(\bar{M}^l),$$

where \widehat{C}_a is the special closed relation obtained from the simple closed $C_a = \{\bar{x}' = a\} \wedge T_1 \wedge T_2$. This S'_a is irreducible by Corollary 5.2.12, and it has dimension $= \dim(S_a)$ and is a subset of S_a . We can see that the dimensions are equal when we notice that $\dim(pr^{-1}(a) \cap \widehat{T}_1 \cap \widehat{T}_2) = \dim(\widehat{C}_a)$. If there are no equational ideals, F , such that $S_a \cap F \subsetneq S_a$ and $\dim(S_a \cap F) = \dim(S_a)$ then $pr^{-1}(a) \cap \widehat{T}_1 \cap \widehat{T}_2 = \widehat{C}_a$ so that $S_a = S'_a$, so S_a is irreducible and we are done. Otherwise suppose that F_1, \dots, F_m are the minimal equational ideals with this property. Then the sets

$$S_a^i = \widehat{C}_a^i(M^{n-l}) \times (I_1 \wedge I_2)(\bar{M}^l),$$

where \widehat{C}_a^i is the special closed relation obtained from the simple closed $C_a^i = C_a \wedge F_i = \{\bar{x}' = a\} \wedge T_1 \wedge T_2 \wedge F_i$, are irreducible and by choice of F_i have $\dim(S_a^i) = \dim(S_a)$ and $S_a = \bigcup_{i=1}^m S_a^i \cup S'_a$. Thus our claim is proved and with it the theorem. \square

5.5. Analytic rank

With reference to [16] we introduce the notion of analytic rank to our canonical model in order to extend the analogy with genuine analytic spaces. Given $U \subseteq_{op} \bar{M}^n$ we define the analytic rank in U of a set $S \subseteq_{cl} U$ exactly as in Section 2.1.

Notation 5.5.1. If $S \subseteq_{cl} \bar{M}^n$ and $U \subseteq_{op} \bar{M}^n$ but $S \not\subseteq U$ we write $\text{ark}_U(S)$ to mean $\text{ark}_U(S \cap U)$.

Note 5.5.2. $\text{ark}_U(S) = 1$ iff S is analytic in U and non-empty.

Proof. If $S \subseteq_{an} U$ then take $S_1 = \emptyset$ to get $\text{ark}_U(S_1) \leq 0$, and $S_2 = S$ analytic in $U \setminus S_1 = U$, so that $\text{ark}_U(S) \leq 1$, and since S is non-empty, $\text{ark}_U(S) \geq 1$.

If $\text{ark}_U(S) \leq 1$ then there is $S_1 \subseteq S$ closed in U such that $\text{ark}_U(S_1) \leq 0$ and $S \setminus S_1$ is analytic in $U \setminus S_1$. But this S_1 must then be empty, and so $S \setminus S_1 = S$ is analytic in $U \setminus S_1 = U$. \square

Note 5.5.3. ark_U measures in some way how far from being analytic in U a set $S \subseteq_{cl} U$ is. S_1 can be seen as containing the points of S at which it is furthest from being analytic in U (i.e. the ‘worst’ points) so that when we take them away we are left with an analytic set. If we can simply find a subset S_1 of S such that $S_1 \subseteq_{an} U$ and $S \setminus S_1 = S_2 \subseteq_{an} U \setminus S_1$. Then S has analytic rank ≤ 2 in U .

Since for any locally \mathcal{L}^* -closed S and any $U \subseteq_{op} \bar{M}^n$ we have $S \cap U \subseteq_{an} U$ by 5.2.3 we immediately get that all non-empty sets locally \mathcal{L}^* -closed have analytic rank 1 in any $U \subseteq_{op} \bar{M}^n$. Thus we find analytic rank most interesting when considering it in open sets U with $U \cap (\bar{M}^n \setminus \bar{M}^n) \neq \emptyset$. With reference to our complex analytic prototype this is where the essential singularities occur.

Theorem 5.5.4. Given $U \subseteq_{op} \bar{M}^n$, any S which is \mathcal{L}^* -closed in U can be given analytic rank in U .

The proof relies on the following results.

Note 5.5.5. If $U \subseteq_{op} \bar{M}^n$ and $C \subseteq \bar{M}^n$ is defined by equalities alone then $C \cap U \subseteq_{an} U$ and so $\text{ark}_U(C \cap U) = 1$.

Proposition 5.5.6. Given $U \subseteq_{op} \bar{M}^n$ and special closed $C \subseteq \bar{M}^n$ we have $\text{ark}_U(C \cap U) \leq 2$.

Proof. By the definition of special closed sets $C = C(M) \cup (\bar{M}^n \setminus M^n)$ for $C(M)$ either a main part of a simple closed set or of dimension 0. We let $S = C \cap U$, and $S_1 = S \cap (\bar{M}^n \setminus M^n) = U \cap (\bar{M}^n \setminus M^n)$. Now since $\bar{M}^n \setminus M^n = \{\bar{x} \in \bar{M}^n \mid \bigvee_{i=1}^n x_i = \infty\}$ is defined by equality alone $\text{ark}_U(U \cap (\bar{M}^n \setminus M^n)) = \text{ark}_U(S_1) = 1 \leq 1$. Further $S_2 = S \setminus S_1 = S \cap M^n = C \cap (U \cap M^n)$ is \mathcal{L}^* -closed in $U \cap M^n$ and so analytic there by Corollary 5.2.4. Thus $\text{ark}_U(S) = \text{ark}_U(C \cap U) \leq 2$. \square

Proposition 5.5.7. Let $C, S \subseteq_{cl} U$. If $\text{ark}_U(C) \leq n$ and $\text{ark}_U(S) \leq m$ then

1. $\text{ark}_U(C \cup S) \leq \max\{n, m\}$;
2. $\text{ark}_U(C \cap S) \leq n.m$.

Proof. See [16]. \square

Proposition 5.5.8. Let $C \subseteq_{cl} U$ and $S \subseteq_{cl} V$. If $\text{ark}_U(C) \leq n$ and $\text{ark}_V(S) \leq m$ then $\text{ark}_{U \times V}(C \times S) \leq n.m$.

Proof. See [16]. \square

Proof of Theorem. Given $U \subseteq_{op} \bar{M}^n$ and $S \subseteq U$ is \mathcal{L}^* -closed in U we have that $S = S' \cap U$ for some \mathcal{L}^* -closed $S' \subseteq \bar{M}^n$. By Proposition 4.2.13 this S' is positively quantifier free definable. The propositions above immediately give us the result that if S is positively quantifier free definable in \mathcal{L}^* then $S \cap U$ can be given analytic rank in U . \square

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